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THE ELEMENTARY

GEOMETRY OF CONICS.

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THE ELEMENTARY

GEOMETRY OF CONICS.

BY

C. TAYLOR, M.A.

FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

THIRD EDITION.

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PREFACE.

THE Geometry of Conics has been recast in the Third Edition, so as to serve as an introduction to a larger work now approaching completion.

The characteristic feature of the edition is the use of the Eccentric Circle, which contributes to a concise and uniform treatment of the three species of conics.

The Asymptotes of the hyperbola are shewn to be coincident with its self-conjugate diameters, and their properties are deduced from a limiting case of a property of conjugate diameters in general.

The principle that Chord-properties should be proved independently of Tangent-properties is still adhered to, although in the general rearrangement of the text it seemed no longer desirable to confine the two classes of properties to separate chapters.

The work now to some extent resembles my first work on *Geometrical Conics*, published in 1863; but the general chapter has been made more complete than I was then able to make it.

C. TAYLOR.

ST JOHN'S COLLEGE,
October, 1879.

CONTENTS.

CHAPTER PAGE

INTRODUCTION 1

DEFINITIONS 6

I. DESCRIPTION OF THE CURVE 9

II. THE GENERAL CONIC 14

III. THE PARABOLA 28

IV. CENTRAL CONICS 41

V. THE ASYMPTOTES 62

VI. THE EQUILATERAL HYPERBOLA 68

VII. THE CONE 72

VIII. ORTHOGONAL PROJECTION 78

EXAMPLES 85

THE GEOMETRY OF CONICS.

INTRODUCTION.

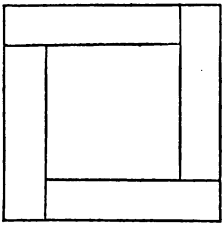
WE shall have occasion in the course of the work to assume the following

LEMMAS.

A. *To prove geometrically that*

$$(a+b)^2-(a\sim b)^2=4ab.$$

If four rectangles whose sides are equal to a and b be fitted symmetrically about the square on $a\sim b$, the whole figure will make up the square on $a+b$.



Therefore $\qquad (a+b)^2=(a\sim b)^2+4ab,$

or $\qquad (a+b)^2-(a\sim b)^2=4ab.$

The same is proved in Euclid II. 8, but by an unsymmetrical construction which shews only a gnomon of equal area instead of the four rectangles.

In illustration of the above, let PQ be any straight line bisected in O, and let Y be any point in PQ produced (figure Art. 13); then

$$\begin{aligned} PY^2 - QY^2 &= (OY + OP)^2 - (OY - OP)^2 \\ &= 4OY\,.\,OP = 2OY\,.\,PQ^{*}. \end{aligned}$$

B. *The distance of any point in a straight line from the middle point of the line is half the sum or difference of its distances from the extremities of the line.*

For let mM be any straight line and L its middle point. And first let a point R be taken in mM produced. Then

$$RL - RM = \tfrac{1}{2}\,mM = Rm - RL,$$

or

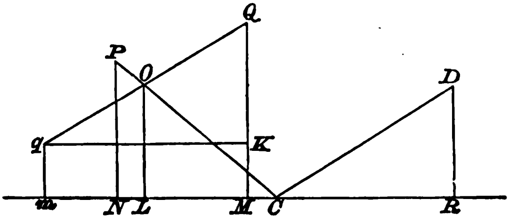
$$RL = \tfrac{1}{2}\,(RM + Rm).$$

Next let a point N be taken within mM. Then

$$NM - NL = \tfrac{1}{2}\,mM = Nm + NL,$$

or

$$NL = \tfrac{1}{2}\,(NM \sim Nm).$$



C. *The ordinate of the middle point of a straight line is equal to half the sum or difference of the ordinates of its extremities.*

From the extremities of a straight line Qq and from its middle point O let parallels QM, qm, OL be drawn to meet any other given straight line; and let these parallels be called the *ordinates* of the points from which they are drawn.

---

* Otherwise thus. Since $PY + QY = 2OY$ (Lemma B), and $PY - QY = PQ$; therefore (Euc. II. 5, Cor.) $PY^2 - QY^2 = 2OY\,.\,PQ$.

Draw qK parallel to mM to meet QM. Then, since qK cuts off from OL a length equal to $\frac{1}{2}QK$, therefore

$$OL - qm = \tfrac{1}{2}(QM - qm),$$

or

$$OL = \tfrac{1}{2}(QM + qm).$$

If the line mM had been drawn so as to divide Qq internally, it would have appeared in like manner that

$$OL = \tfrac{1}{2}(QM \sim qm).$$

D. *The sum of the squares of the distances of any point from the extremities of any straight line is double of the sum of the squares of its distance from the middle point of the line and of half the line.*

For if SS' be the given straight line, P the given point, PN a perpendicular to SS', and C the middle point of SS',

then $$SP^2 = CS^2 + CP^2 + 2CS \,.\, CN,$$

and $$S'P^2 = CS^2 + CP^2 - 2CS' \,.\, CN;$$

therefore by addition, since CS' is equal to CS,

$$SP^2 + S'P^2 = 2CS^2 + 2CP^2.$$

E. *To divide a given straight line in a given ratio of majority or of minority.*

A ratio is said to be a ratio of equality, majority or minority according as it is equal to unity, or greater or less than unity.

(i) First let SX be the given straight line, which is to be divided in a given ratio of majority. Draw SH in any direction, and produce it to K, so that $SH : HK$ may be equal to the given ratio. Join KX, and draw HA parallel to KX to meet SX in A. Then since

$$SA : AX = SH : HK, \qquad \text{[Euc. VI. 2.}$$

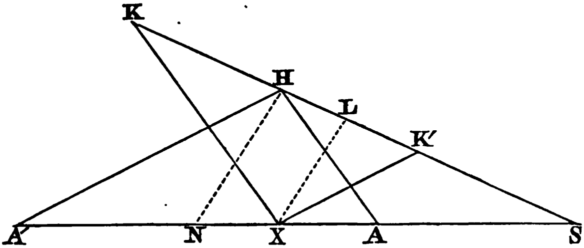
the point A divides SX internally in the given ratio.

1—2

Upon HS take HK' equal to HK. Join $K'X$, and draw HA' parallel to it to meet SX produced in A'. Then

$$SA' : A'X = SH : HK' = SH : HK,$$

or the point A' divides SX externally in the given ratio.



In this case the points A and A' will always lie on the *same side* of S, because K and K' lie on the same side of S.

(ii) Next let XS be the given line, which is to be divided from the end X in a given ratio of minority, viz. that of HK to HS. Using the same lines of construction as before, we determine the two points of division A and A', which must always lie on *opposite sides* of X, because K and K' lie on opposite sides of H.

It is evident that there is only one point at a finite distance which bisects a straight line, or divides it in a ratio of *equality*.

F. *To divide a straight line in a ratio less than a given ratio.*

In the second case of Lemma E take a point N at random between A and A'; and (supposing for example that N lies between X and A') join NH, and draw XL parallel to it to meet HK' in L. Then

$$\begin{aligned} XN : NS = LH : HS &< K'H : HS, \\ &< XA' : A'S. \end{aligned}$$

Hence any point N between A and A' divides XS in a ratio less than the ratio of *minority* $XA : AS$, as was to be done; and it may be seen that no other point so divides the given line XS.

If SX be the line to be divided from the end S in a ratio less than the ratio of *majority* $SA : AX$, it may be shewn similarly that the point of division is to be taken in AA' produced either way, and *not* between A and A'.

G. *Harmonic section of a straight line.*

If a straight line SX be divided internally and externally in the same ratio at A and A' so that

$$SA' : A'X = SA : AX;$$

then

$$\begin{aligned} SA' : SA &= A'X : AX \\ &= SA' \sim SX : SX \sim SA, \end{aligned}$$

or SA', SX, SA are in harmonic progression.

Hence SX is said to be divided harmonically at A and A'.

DEFINITIONS.

[Further definitions will be given at the beginnings of the chapters, as occasion arises.]

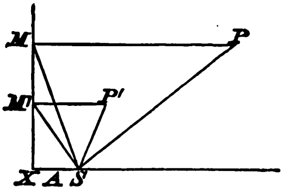
1. A CONIC SECTION*, or briefly a *Conic*, is the curve traced in a plane by a point which moves in such a way that its distance from a given point is in a constant ratio to its distance from a given straight line. The given point is called the *Focus*, the given straight line the *Directrix*, and the constant ratio the *Eccentricity* of the conic.

If S be the focus, P any point on the conic and PM the perpendicular from it to the directrix, the ratio of SP to PM is constant. If P' be any other point on the conic, and $P'M'$ the perpendicular from it to the directrix, then

$$SP : PM = SP' : P'M',$$

or

$$SP : SP' = PM : P'M'.$$



Let us now take a particular case, and suppose the directrix to be at an infinite distance from the focus. In this case $PM : P'M'$ is a ratio of equality, and therefore $SP : SP'$ is a ratio of equality. That is to say, *SP* is always equal to *SP′*, and the locus of *P′* is a circle. Thus it appears that our definition of a conic is an extension of the definition of a circle.

---

* The conic sections were so called because they are the curves in which a plane can be made to intersect a cone, as will be shewn in the seventh chapter.

2. A conic is called a *Parabola*, an *Ellipse*, or a *Hyperbola*, according as its eccentricity is a ratio of equality, of minority, or of majority.

3. The *Axis* is the unlimited straight line through the focus at right angles to the directrix, and the points in which it meets the conic are called the *Vertices*. When one vertex only is spoken of the vertex which lies between the focus and the directrix is signified.

It is evident from Lemma E that the parabola has only one vertex at a finite distance, and that the ellipse and the hyperbola have each two vertices.

4. The middle point of the line joining the vertices is called the *Centre* of the conic. The ellipse and the hyperbola are called *Central Conics*, in contrast with the parabola which has no centre at a finite distance. The straight line through the centre at right angles to the axis is called the *Conjugate Axis*.

5. A *Chord* of a conic is properly the finite straight line joining any two points on the curve; but the term is also used to denote the unlimited straight line joining any two points on the curve. The extremities of a chord are the points in which it meets the conic.

6. The *Latus Rectum* is the focal chord, or chord through the focus, at right angles to the axis.

7. A *Diameter* is the locus of the middle points of a system of parallel chords: it will be proved that the diameters of conics are straight lines. One diameter is said to be *conjugate* to another when it bisects chords parallel thereto.

8. The *Principal Ordinate*, or briefly the *Ordinate*, of any point is the perpendicular drawn from it to the axis. More generally, the ordinate of any point *to any diameter* is

the line drawn from the point to that diameter in the direction parallel to the conjugate diameter.

9. A *Tangent* to a conic is the limiting position of a chord or secant whose two points of intersection with the curve have become coincident. Thus if P and Q be adjacent points on a conic, and if the chord joining them be turned round P, or be moved about in any other way, until its extremity Q coincides with P, the chord in its limiting position becomes the tangent at P. Hence a tangent is said to be a straight line which passes through two *consecutive* or *coincident* points on the curve.

The chord of contact of two tangents is the chord joining their points of contact.

10. The *Normal* at any point of a conic is the perpendicular to the tangent at that point.

11. If about any point in the plane of a conic other than the centre of the conic a circle be described, such that the ratio of its radius to the perpendicular distance of its centre from the directrix is equal to the eccentricity, the circle may be called the eccentric circle of the conic with respect to that point, or briefly the *Eccentric Circle of the Point.*

12. The *Order* or *Degree* of a curve is determined by the number of points in which it can be met by a straight line. Thus a curve of the second order or degree is one which a straight line meets generally in two and never in more than two points.

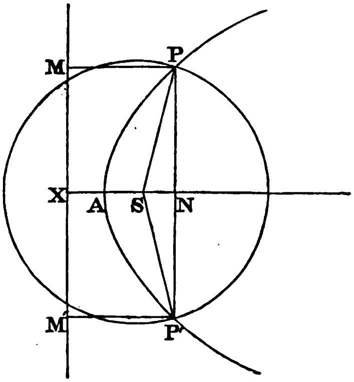
CHAPTER I.

DESCRIPTION OF THE CURVE.

1. *Having given the focus, directrix and eccentricity of a conic, it is required to describe the curve.*

Let S be the focus*, MM' the directrix, and X the point in which the axis meets the directrix. In SX take the point A so that the ratio of SA to AX may be equal to the eccentricity (Lemma E). Then A is the vertex of the conic.

Draw a straight line cutting the axis at right angles in N; and let P and P' be the points in which the line meets



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* The planets describe approximately ellipses about the sun in one focus. For this reason the first letter of *Sol* is used, as by Newton, to denote the *Focus*, or as he called it the Umbilicus. We shall use the letters S, A, X as above without further explanation, so that $SA : AX$ will always denote the eccentricity.

the circle described with S as centre and radius SP, such that

$$SP : NX = SA : AX.$$

Draw PM perpendicular to the directrix. Then

$$SP : PM = SP : NX = SA : AX,$$

or P is a point on the conic. In like manner it may be shewn that P' is a point on the conic.

If now we suppose the chord PP' to slide at right angles to the axis, so as to assume all *possible* positions, its extremities will trace out the complete curve.

2. *The three species of conics.*

In order that the line and the circle in the above construction may intersect, the length SN must be less than the radius SP, or

$$SN : NX < SA : AX.$$

In the case of the *Parabola* we must have $SN < NX$. The point N may therefore be taken anywhere in XA produced, and the curve consists of one infinite branch spreading out from the vertex and away from the directrix.

In the *Ellipse*, if A' be the second vertex, the point N may be taken anywhere between A and A' (Lemma F), and the curve consists of one oval branch (fig. Art. 5) lying on the same side of the directrix with the focus.

In the *Hyperbola*, if A' be the second vertex, the point N may be taken anywhere in AA' produced (Lemma F), and the curve consists of two infinite branches on opposite sides of the directrix, placed back to back, as in Figure 4.

3. *The symmetry of the curve.*

From the foregoing construction it is evident that the curve is symmetrical with respect to its axis, since its points are always determined in pairs as P and P' in corresponding positions above and below the axis, so that the part of the curve below the axis is the accurate reflexion of the part above the axis.

It follows also that the tangent at A is at right angles to the axis, since when the point N coincides with the vertex, $SP = SA = SP'$; that is to say, the points P and P' coalesce at A, and the chord joining them, which is always at right angles to the axis, becomes the tangent to the conic at its vertex (Def. 9).

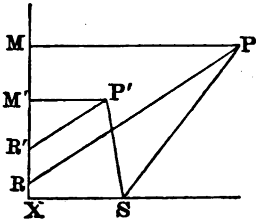
4. *The focal distance of any point on a conic is in a constant ratio to the distance of the point from the directrix measured parallel to any fixed straight line which meets the directrix.*

From any two points P and P' on the conic draw PR and $P'R'$ in any fixed direction to meet the directrix, and draw PM and $P'M'$ perpendicular to the directrix.

Then $\quad SP : PM = SP' : P'M',$ [Def. 1.

and $\quad PM : PR = P'M' : P'R',$

by similar triangles.



Therefore $\quad SP : PR = SP' : P'R',$

or (since we may consider P' and $P'R'$ to remain fixed whilst P varies) the focal distance SP varies as the distance PR to the directrix measured in any given direction.

Conversely, every point P which satisfies the above relation is a point on the conic.

Notice in particular that if any chord PQ meet the directrix in R,

$$SP : PR = SQ : QR.$$

5. *A conic is a curve of the second order.*

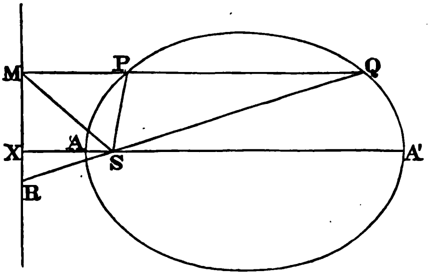
Let a straight line drawn in any direction cut the conic in P and meet the directrix in M. Make the angle MSR equal to MSP, and let the line SR produced either way meet MP in Q.

Then, by Euclid VI. A or 3, since SM bisects the angle PSR or its supplement,

$$SQ : SP = QM : PM,$$

or

$$SQ : QM = SP : PM;$$



and therefore Q is a point on the curve (Art. 4); and it is evident that no third point of intersection of the line PQ with the conic can be determined.

It follows that a straight line which meets a conic will in general meet it in two points, and never in more than two. A conic is therefore a curve of the second order or degree (Def. 12).

If PM be supposed parallel to the axis and equal to SP, the line RS will likewise be parallel to the axis. Hence every straight line MP drawn parallel to the axis of a *Parabola* meets the curve in one point only.

6. *To describe a conic of given focus, directrix and eccentricity by means of the eccentric circle of any assumed point.*

Describe the eccentric circle of any point O* in the plane of the conic (Def. 11), and let a straight line through S meet the circle in p and the directrix in R.

---

* Let SL be the semi-latus rectum, and let the ordinate of O meet the axis in N and XL in K. Then since $KN : OD = KN : NX = SL : SX = SA : AX$, the radius of the circle must be taken equal to KN.

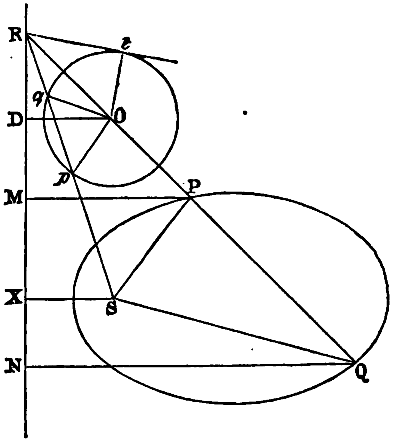
Let the focal radius parallel to pO meet RO in P, and let OD and PM be perpendiculars to the directrix.

Then since SP, pO and PM, OD are parallels,

$$\begin{aligned} SP : Op &= PR : OR \\ &= PM : OD, \end{aligned}$$

or

$$\begin{aligned} SP : PM &= Op : OD \\ &= \text{the eccentricity.} \end{aligned}$$



Hence, as p moves round the circle, P traces the conic which was to be described.

Conversely, if any point O be taken on a chord PQ of a conic, the eccentric circle of O will meet SR (drawn to the point of concourse R of the chord with the directrix) in points p and q lying upon radii parallel to SP and SQ.

CHAPTER II.

THE GENERAL CONIC.

We shall commence by proving some of the principal properties which are common to the parabola, the ellipse and the hyperbola.

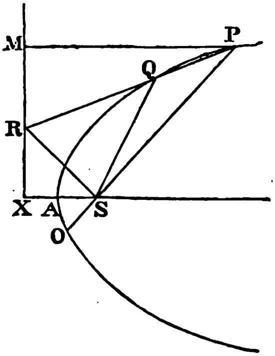
The Tangent.

PROPOSITION I.

7. *Each of the two tangents which can be drawn to a conic from any point on its directrix subtends a right angle at the focus.*

Let P and Q be adjacent points on the curve, and let PQ produced meet the directrix in R. Then it may be shewn (Art. 4) that

$$SP : SQ = PR : QR;$$



and therefore SR bisects the angle which SQ makes with PS produced. [Euc. VI. A.

Let PS meet the curve again in O. Then since the angles RSQ and RSO are always equal, therefore in the limit, when Q coalesces with P, each of them becomes a right angle, and RP, which becomes the tangent at P (Def. 9) subtends a right angle at S. In like manner it may be shewn that RO, which is the tangent at O, subtends a right angle at S.

Corollary.

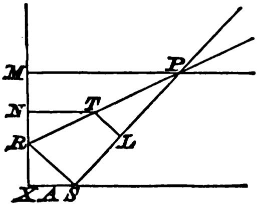
Hence it appears that the tangents at the extremities of any focal chord OP meet at a point R lying on the directrix, and such that SR is at right angles to OP. Conversely, if tangents be drawn to a conic from any point R on the directrix their chord of contact will be the focal chord at right angles to SR. The tangents at the extremities of the latus rectum meet at X.

PROPOSITION II.

8. *If from any point* T *on the tangent at* P *there be drawn perpendiculars* TL *and* TN *to* SP *and the directrix, the ratio of* SL *to* TN *will be constant and equal to the eccentricity:*

For if the tangent at P meet the directrix in R, and if PM be a perpendicular to the directrix, then since SR is at right angles to SP (Prop. I.) and is therefore parallel to TL, it follows that

$$SL : SP = TR : PR$$
$$= TN : PM.$$



Therefore $$SL : TN = SP : PM$$
$$= SA : AX.$$

The line SL is equal to the radius of the eccentric circle of T.

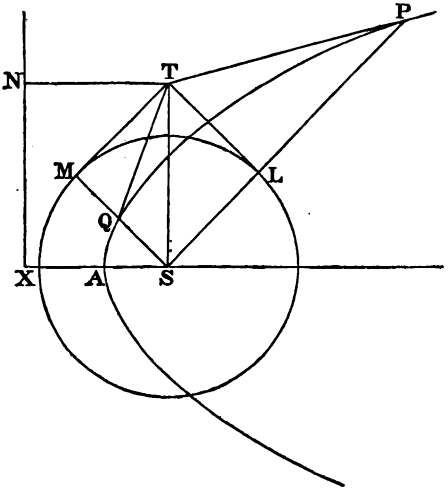
Corollary.

To draw a pair of tangents to a conic from a given external point T, with S as centre describe a circle equal to the eccentric circle of T, and draw the tangents TL and TM to the circle; then by the converse of the proposition SL and SM will pass through the points of contact of the required tangents. Draw SR at right angles to SL to meet the directrix in R; then TR is one of the two tangents. Draw SR' at right angles to SM to meet the directrix in R'; then TR' is the second tangent from T.

PROPOSITION III.

9. *The two tangents which can be drawn to a conic from any external point subtend equal or supplementary angles at the focus.*

For if TP and TQ be the two tangents to a conic from



the point T, and TL, TM, TN be perpendiculars to SP, SQ and the directrix, then by Prop. II., since T lies on the tangent at P,

$$SL : TN = SA : AX;$$

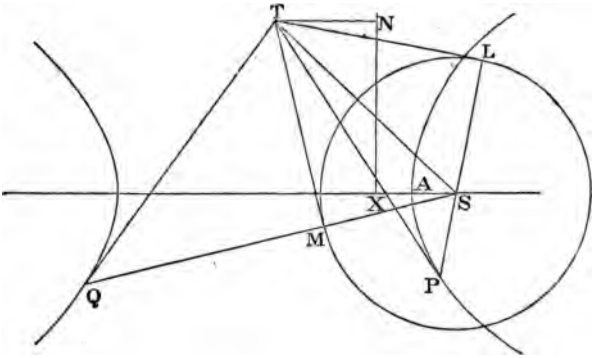
and since T lies on the tangent at Q,

$$SM : TN = SA : AX.$$

Therefore in the right-angled triangles STL and STM the side SL is equal to SM; and the hypotenuse ST is common; and therefore the angle TSL is equal to TSM.

Now (i) if TP and TQ touch the same branch of the conic, the angles which they subtend at S are either *equal* to TSL and TSM (as in the above figure) or *supplementary* thereto. In either case the two tangents subtend EQUAL angles at S.

But (ii) if TP and TQ touch opposite branches of a hyperbola (as in the next figure), so that one and one only of the lines SL and SM has to be produced backwards to P or Q, the two tangents will subtend SUPPLEMENTARY angles at S.

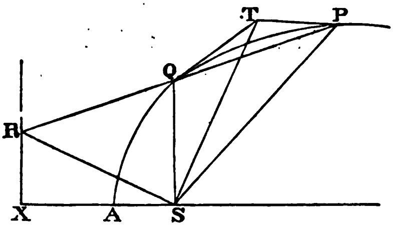


PROPOSITION IV.

10. *The point of concourse of any two tangents to a conic and the point in which their chord of contact meets the di-*

rectrix lie upon a pair of focal radii which include a right angle.

If P and Q be points on the same branch of a conic, and if PQ meet the directrix in R, then, as in Prop. I, SR bisects



the angle which SQ makes with PS produced; that is to say, it bisects the supplement of PSQ.

Also, if the tangents at P and Q meet in T, the line ST bisects the angle PSQ (Prop. III.).

Therefore ST and SR bisect supplementary angles, and are therefore at right angles to one another.

If TP and TQ touch opposite branches of a hyperbola, then (completing the second figure of Art. 9) it may be shewn that in this case also the angle TSR is a right angle.

The Normal.

PROPOSITION V.

11. *The normal at any point of a conic meets the axis at a distance from the focus which is to the focal distance of the point in a constant ratio equal to the eccentricity.*

For if the tangent at P meet the directrix in R, the circle on PR as diameter will pass through S, since PSR is a right angle; and it will likewise pass through M, the projection of P upon the directrix; and the normal at P will touch the circle, since it is at right angles to its diameter PR.

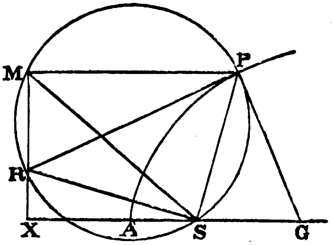
Let the normal meet the axis in G.

Then $\angle SPG = SMP$,

in the alternate segment of the circle, and

$$\angle PSG = SPM,$$

by parallels.



Therefore the triangles SPG and PSM are similar, and

$$SG : SP = SP : PM$$
$$= SA : AX,$$

or SG varies as SP, as was to be proved.

Conversely, if in AS produced there be taken a point G such that

$$SG : SP = SA : AX,$$

the line PG will be the normal at P.

PROPOSITION VI.

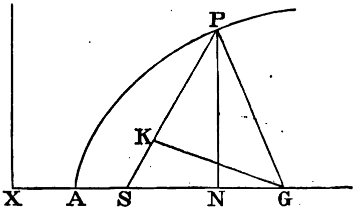
12. *At any point of a conic, the projection of the normal (terminated by the axis) upon the focal radius is equal to the semi-latus rectum.*

Let the normal at P meet the axis in G: draw GK perpendicular to SP: and draw PN perpendicular to the axis.

Then by similar right-angled triangles SKG and SNP,

$$SK : SN = SG : SP$$
$$= SP : PM \qquad \text{[Prop. v.}$$
$$= SP : NX.$$

Hence $SP - SK : NX - SN = SP : NX$;



that is to say, PK is to SX in a constant ratio equal to the eccentricity, and is therefore equal to the semi-latus rectum.

Note that when P is taken at an extremity of the latus rectum PK coalesces with the semi-latus rectum.

Diameters.

PROPOSITION VII.

13. *The locus of the middle points of any system of parallel chords of a conic is a straight line which meets the directrix on the straight line through the focus at right angles to the chords.*

Let PQ be any one of a system of parallel chords, V the point in which the focal perpendicular upon them meets the directrix, R and Y the points in which PQ meets the directrix and SV respectively.

Then since $SP : PR = SQ : QR$, [Art. 4.

therefore (supposing for example that SP is greater than SQ)

$$SP^2 - SQ^2 : PR^2 - QR^2 = SP^2 : PR^2;$$

or, subtracting SY^2 from each of the first two squares,

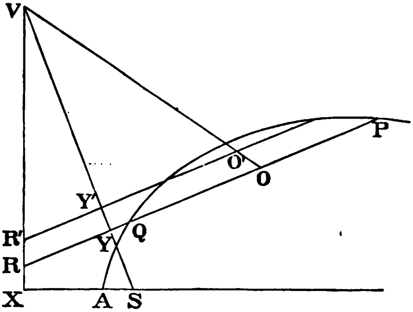
$$PY^2 - QY^2 : PR^2 - QR^2 = SP^2 : PR^2.$$

[Euc. I. 47.

But if O be the middle point of PQ,

$$PY^2 - QY^2 = 2OY.PQ; \qquad \text{[Lemma A.}$$

and similarly, $$PR^2 - QR^2 = 2OR.PQ.$$



Hence $$OY : OR = SP^2 : PR^2,$$

which (Art. 4) is a *constant ratio* so long as PQ is drawn in a specified direction.

Hence if any other chord be drawn parallel to PQ, and if O', Y', R' be the new positions of O, Y, R, it follows that

$$O'Y' : O'R' = OY : OR;$$

and hence that the points O, O', V lie in a straight line*.

If now we suppose the point O' to remain fixed whilst PQ moves parallel to itself, the point O will always lie upon a *fixed* straight line $O'V$, as was to be proved.

Corollary 1.

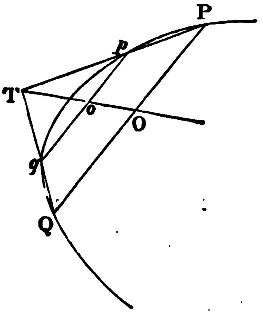
The tangent at either extremity of a diameter is parallel to the chords which the diameter bisects, since any one of the bisected chords may be supposed to move parallel to itself until its segments vanish together and its extremities coalesce, so that it becomes a tangent, viz. at an extremity of the bisecting diameter. If a diameter meets the conic in two points, the tangents at those points are parallel to the *ordinates* (Def. 8) of that diameter and to one another.

---

* For if they do lie on a straight line, $O'Y' : OY = O'V : OV = O'R' : OR$. The required converse may be easily deduced by a *reductio ad absurdum*.

Corollary 2.

If PQ and pq be any two of a system of parallel chords, and O and o be their middle points, which will lie on a fixed diameter, it is evident that Pp and Qq will meet at a point T lying on that diameter. Hence, making pq move parallel to itself until it coalesces with PQ, so that TP and TQ become the tangents at P and Q, we see that *the tangents at the extremities of any chord meet upon the diameter which bisects the chord;* and conversely, that the diameter to any external point bisects the chord of contact of the two tangents from that point.



PROPOSITION VIII.

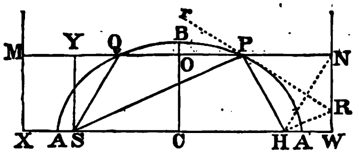
14. *Every central conic is divided symmetrically by its conjugate axis, and has a second focus and directrix.*

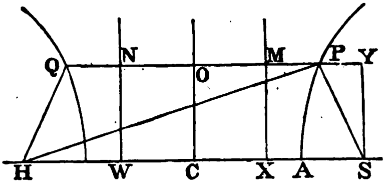
Let AA' be the transverse axis of a central conic, PQ any chord parallel thereto, and M the point in which PQ or its prolongation meets the directrix.

Bisect PQ in O, and draw SY perpendicular to PQ; then it may be shewn, precisely as in Prop. VII., that

$$OY : OM = SP^2 : PM^2;$$

and hence that the locus of O is a straight line at right angles to AA'.





Since this straight line must also bisect AA' (which is a limiting position of the chord PQ), it meets AA' at the centre C of the conic, and coincides with the *conjugate axis* (Def. 4).

It is evident that this line divides the curve into two parts such that each is the exact reflexion of the other; and hence that the curve has a second focus H and directrix NW, the exact counterparts of the original focus and directrix with reference to which the conic was defined.

Corollary 1.

From the symmetry of a central conic with respect to its two axes, it is manifest that every chord through its centre is bisected at that point, and hence that all diameters pass through the centre. It is further evident that any two diameters or focal chords equally inclined to either axis are equal to one another; and that any two tangents to the conic from a point on either axis are likewise equal.

Corollary 2.

In Art. 13 let the diameter parallel to PQ meet the directrix in V', so that VS is at right angles to CV', and S is the orthocentre of the triangle CVV'. Hence, the diameter CV' bisects chords parallel to CV, or if one diameter be conjugate to a second, the second is conjugate to the first.

The Segments of Chords.

PROPOSITION IX.

15. *The semi-latus rectum is a harmonic mean between the segments of any focal chord.*

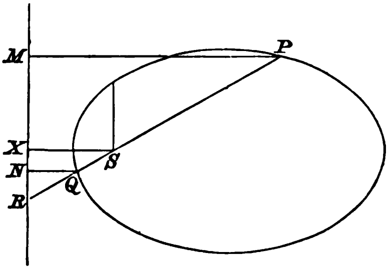
Let a focal chord PQ or its prolongation meet the directrix in R, and let PM, SX, QN be perpendiculars to the directrix.

Then, since

$$SP : SQ = PM : QN = PR : QR,$$

therefore $\quad PR - SR : SR - QR = PR : QR,$

or PR, SR, QR are in harmonic progression. [Lemma G.



But by parallels, and from the definition of the curve, if L be used to denote the semi-latus rectum,

$$\begin{aligned} PR : SR : QR &= PM : SX : QN \\ &= SP : L : SQ. \end{aligned}$$

Therefore also SP, L, SQ are in harmonic progression.

Corollary.

It is easy to deduce that

$$L \,.\, PQ = L\,(SP + SQ) = 2SP \,.\, SQ.$$

Hence, if pq be any other focal chord,

$$PQ : pq = SP \cdot SQ : Sp \cdot Sq,$$

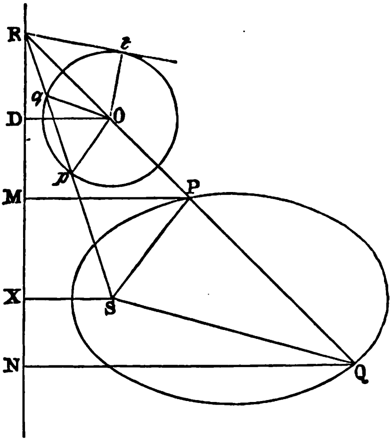
or *focal chords are to one another as the rectangles contained by their segments.*

PROPOSITION X.

16. *A chord of a conic being divided at any point, to determine the rectangle contained by its segments.*

Let PQ be any chord of a conic, and O any point on PQ, or PQ produced; it is required to determine the magnitude of the rectangle $OP \cdot OQ$.

Let the chord meet the directrix in R, and let OD be a perpendicular to the directrix. Draw the eccentric circle of O, and let it cut SR in p and q. Then it may be shewn that Op is parallel to PS and Oq to QS. [Art. 6.



Therefore $\qquad OP : Sp = OR : Rp,$

and $\qquad OQ : Sq = OR : Rq.$

$$\text{Hence } OP.OQ : Sp.Sq = OR^2 : Rp.Rq$$
$$= OR^2 : Rt^2,$$

if Rt be one of the tangents from R to the circle*.
[Euc. III. 36.

In this result it is to be noticed (i) that the magnitude of $Sp.Sq$ depends only upon the *position* of O, since when O is given its eccentric circle is given, and $Sp.Sq$ is constant; and (ii) that the ratio $OR^2 : Rt^2$ depends only upon the *direction* of PQ, since when the angle ORD is given, OR varies as OD, and therefore (by the definition of the eccentric circle) as Ot, and therefore OR^2 varies as $OR^2 \sim Ot^2$, or as Rt^2.

Corollary 1.

If through any other point O' there be drawn a chord $P'Q'$ parallel to PQ, and if p' and q' be the corresponding positions of p and q, viz. on the eccentric circle of O', it follows (since the ratio $OR^2 : Rt^2$ depends only upon the *direction* of the chord) that

$$OP.OQ : Sp.Sq = O'P'.O'Q' : Sp'.Sq',$$

where the consequents depend only upon the *positions* of O and O'. If therefore any second pair of parallel chords be drawn through the same point, we have the general theorem that:

The ratio of the rectangles contained by the segments of any two intersecting chords of a conic is the same as for any other two chords parallel to the former, each to each.

Corollary 2.

This ratio is equal to that of the parallel focal chords (Art. 15, Cor.); and in a central conic to that of the squares of the semi-diameters parallel to the chords; and in the general conic to the ratio of the squares of any pair of tangents parallel to the chords, since a tangent is defined as a chord whose extremities are coincident.

---

* When P and Q lie on opposite branches of a hyperbola the point R falls within the circle. In this case draw Rt the semichord at right angles to OR in the circle, and apply Euclid III. 35.

Corollary 3.

If a circle and a conic intersect in four points, their common chords are equally inclined, in opposite pairs, to the axis of the conic. For if POQ and pOq be one of the three pairs of common chords of a circle and a conic, the rectangles $PO \cdot OQ$ and $pO \cdot Oq$ are as the focal chords parallel to PQ and pq (Cor. 2); and the same rectangles are equal to one another by a property of the circle. Therefore the focal chords parallel to PQ and pq are equal, and are therefore equally inclined to the axis. [Art. 14, Cor. 1.

CHAPTER III.

THE PARABOLA.

17. THE annexed further definitions will be required in the present chapter.

The principal *Abscissa* or *Absciss* of any point with respect to a parabola is the finite segment cut off from the axis by the principal ordinate of the point. The abscissa or absciss of a point *to any diameter* is the finite segment cut off from it by the ordinate of the point to that diameter. [Def. 8.

The *Parameter* of any diameter of a parabola is the focal chord which it bisects: thus the latus rectum is the parameter of the axis.

The *Subtangent* at any point of a conic is the intercept upon the axis between the tangent and the ordinate of the point; and the *Subnormal* is the intercept between the normal and the ordinate. The subtangent *to any diameter* is the intercept thereupon between the tangent and the ordinate of its point of contact to that diameter.

18. The eccentricity of the parabola being a ratio of equality, the semi-latus rectum is equal to SX, and therefore to $2SA$. It is to be noted also that in the case of the parabola SG becomes equal to SP (Art. 11), and the eccentric circles of all points (Def. 11) touch the directrix.

The property of diameters in Art. 13 has been proved for all conics without distinction; but we shall also shew that it can be proved with peculiar ease for the special case of the parabola.

Chord-Properties*.

PROPOSITION I.

19. *The principal ordinate of any point on a parabola is a mean proportional to its abscissa and the latus rectum.*

If PN and AN be the principal ordinate and abscissa of any point P on the parabola, we have to shew that

$$PN^2 = 4AS \,.\, AN.$$

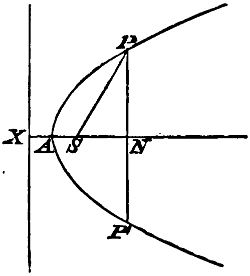
By Euclid I. 47 and from the definition of the parabola,

$$PN^2 + SN^2 = SP^2 = NX^2.$$

Hence (taking for example the case in which AN is greater than AS),

$$\begin{aligned} PN^2 + (AN - AS)^2 &= (AN + AS)^2 \\ &= (AN - AS)^2 + 4AS \,.\, AN. \end{aligned}$$

[Lemma A.



Therefore PN^2 is equal to $4AS \,.\, AN$; or in other words, PN is a mean proportional to the abscissa AN and the latus rectum (Art. 18).

---

* By chord-properties (as distinguished from tangent-properties) we understand such properties as do not presuppose the definition of a tangent (Def. 9). It is desirable to avoid using tangent-properties to prove chord-properties, the reverse order being the only natural one, so long as we regard a tangent as the limit of a chord.

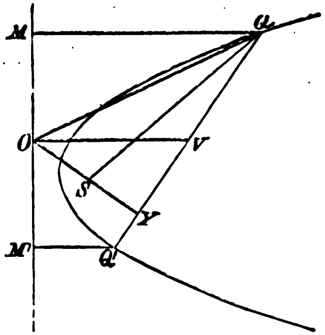
Conversely, if PN and AN be thus related, the locus of P will be a parabola of latus rectum equal to $4AS$.

PROPOSITION II.

20. *The locus of the middle points of any number of parallel chords of a parabola is a straight line parallel to the axis; and the bisecting line meets the directrix on the straight line through the focus at right angles to the common direction of the chords.*

Let QQ' be any one of a system of parallel chords, and let QM and $Q'M'$ be perpendiculars to the directrix.

Let the focal perpendicular upon the chords meet QQ' in Y and the directrix in O; and let the parallel to the axis through O (which is a *fixed* straight line) meet QQ' in V. Then will V be the middle point of QQ'.



For by Euclid I. 47 and from the definition of the parabola (taking the case in which Y lies in OS produced),

$$OM^2 = OQ^2 - QM^2 = OQ^2 - SQ^2$$
$$= OY^2 - SY^2;$$

and OM'^2 may be shewn to have the same value.

And since OM and OM' are thus equal, the line OV parallel to the axis bisects QQ'; that is to say, it bisects every chord which is at right angles to OS.

It is hence evident that every straight line parallel to the axis of a parabola is a diameter (Def. 7) of the curve, and that *all diameters of a parabola are parallel to the axis and to one another.*

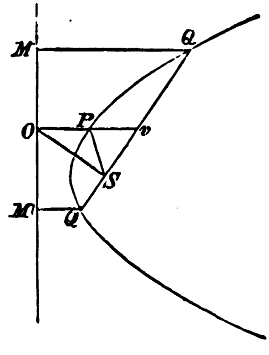
PROPOSITION III.

21. *To evaluate the parameter of any diameter of a parabola.*

Let a diameter meet its parameter QSQ' (Def. Art. 17) in v, the curve in P, and the directrix in O, so that vSO is a right angle. [Prop. II.

Let fall perpendiculars QM and $Q'M'$ upon the directrix. Then

$$QQ' = SQ + SQ' = QM + Q'M'$$
$$= 2vO. \qquad \text{[Lemma C.}$$



And because vSO is a right angle, and $SP = PO$, therefore vO is a diameter of the circle round OSv, and P is its centre.

Hence $QQ' = 2vO = 4SP$,

or the focal chord QQ' is equal to four times the focal distance of the extremity of the diameter of which it is the parameter. In particular, as we have already seen, the latus rectum or principal parameter is equal to $4AS$.

PROPOSITION IV.

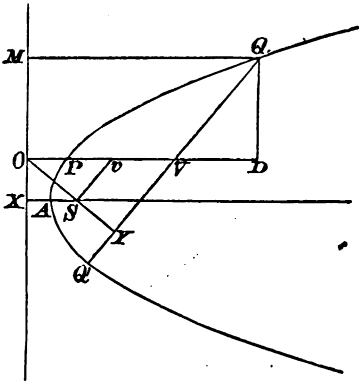
22. *The ordinate of any point on a parabola to any diameter is a mean proportional to the parameter of the diameter and the abscissa of the point.*

Let Q be any point on a parabola, QV and PV its ordinate and abscissa to any diameter: we have to shew that

$$QV^2 = 4SP \, . \, PV,$$

the parameter of the diameter through P being equal to $4SP$. [Prop. III.

Let the diameter meet its parameter in v and the directrix in O; and let OS, which is at right angles to Sv and QV (Prop. II), meet QV in Y.



Then it may be shewn that, if QD and QM be perpendiculars to the diameter and the directrix,

$$QD^2 = OM^2 = OY^2 \sim SY^2. \qquad \text{[Art. 20.}$$

Moreover by parallels,

$$OY : OV = SY : vV; \qquad \text{[Euc. VI. 2.}$$

and therefore (supposing for example that OY is greater than SY)

$$OY^2 - SY^2 : OV^2 - vV = OY^2 : OV^2$$
$$= QD^2 : QV^2,$$

by similar triangles OYV and QDV. And since the antecedents in this proportion are equal, the consequents are equal, so that

$$QV^2 = OV^2 - vV^2.$$

And since P is the centre of the circle round OSv (Art. 21), therefore OV is equal to $PV + SP$, and vV to $PV - SP$; and therefore from above,

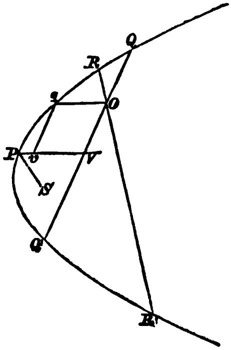
$$QV^2 = (PV + SP)^2 - (PV - SP)^2,$$
$$= 4SP \,.\, PV; \qquad \text{[Lemma A.}$$

that is to say, the ordinate QV is a mean proportional to the abscissa PV and the parameter $4SP$.

PROPOSITION V.

23. *The rectangles contained by the segments of any two intersecting chords are as the parameters of the diameters which bisect them.*

Take any two chords QQ' and RR' intersecting in a point O, within or without the parabola, and let the diameter through O meet the parabola in q.



Bisect QQ' in V, let the diameter through V meet the curve in P, and draw the ordinate qv to that diameter. Then, taking the case in which O lies within the parabola,

$$\begin{aligned} QO\,.\,OQ' &= QV^2 - OV^2, && \text{[Euc. II. 5, Cor.} \\ &= QV^2 - qv^2, \\ &= 4SP\,.\,PV - 4SP\,.\,Pv; && \text{[Prop. IV.} \end{aligned}$$

and therefore, since $PV - Pv$ is equal to vV or qO,

$$QO\,.\,OQ' = 4SP\,.\,qO.$$

Similarly $\quad RO\,.\,OR' = 4SP'\,.\,qO,$

if P' be the extremity of the diameter which bisects RR'; and the same may be proved for the case in which O lies without the parabola.

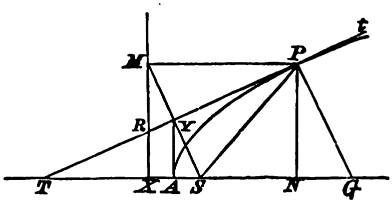
Therefore the rectangles $QO\,.\,OQ'$ and $RO\,.\,OR'$ are as $4SP$ to $4SP'$, or as the parameters of the diameters which bisect QQ' and RR' (Prop. III.), in accordance with what was proved for all conics without distinction in Art. 16.

Tangent-Properties.

PROPOSITION VI.

24. *The tangent to a parabola at any point is the bisector of the angle which the focal distance of the point makes with the diameter produced.*

Let the tangent at P meet the directrix in R, and let the diameter be produced through P without the curve to meet the directrix in M.



Then PSR is a right angle (Art. 7), and $SP = PM$, and PR is common to the right-angled triangles PSR and MPR.

Therefore their angles at P are equal, or the tangent PR is the bisector of the angle SPM.

It is further evident that RP produced makes equal angles with SP and PM; and that if the tangent meet the axis in T, the angles SPT and STP are equal, or the tangent together with SP and the axis determine an isosceles triangle.

Corollary 1.

If PN be the principal ordinate of P, it follows that

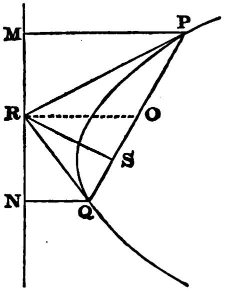
$$ST = SP = NX = AN + AS;$$

and therefore $AN = ST - AS = AT$,

or *the subtangent is double of the abscissa.* [Def. Art. 17.

Corollary 2.

If PQ be any focal chord and MN its projection upon the directrix, and if SR be drawn at right angles to PQ to meet the directrix; it is evident from above that the bisector of the angle SRM is the tangent at P, and that the bisector of SRN is the tangent at Q. Hence



$$\angle PRQ = \tfrac{1}{2}SRM + \tfrac{1}{2}SRN = \text{a right angle},$$

or the tangents at the extremities of any focal chord meet *at right angles* upon the directrix, and conversely.

PROPOSITION VII.

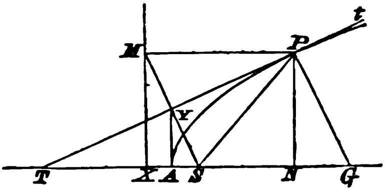
25. *The normal at any point makes equal angles with the focal distance and the axis, and it bisects the interior angle between the focal distance and the diameter to the point.*

If the normal at P meets the axis in G, then $SG=SP$ (Art. 18), and therefore PG makes equal angles with SP and SG.

If the diameter at P be drawn, and any point D be taken upon it *within* the curve, then by parallels and from above,

$$\angle DPG = PGS = GPS,$$

or PG bisects the angle SPD.



Corollary.

If AN be the principal abscissa of P, then since

$$SG = SP = NX,$$

therefore

$$NG = SG - SN = NX - SN = SX,$$

or *the subnormal is equal to semi-latus rectum.* [Def. Art. 17.

PROPOSITION VIII.

26. *The foot of the focal perpendicular to any tangent lies on the tangent at the vertex; and the square of the focal perpendicular to any tangent varies as the focal distance of its point of contact.*

Let the tangent at P meet the axis in T, and let PN be the ordinate of P.

(i) Draw the tangent at the vertex A, and let it meet PT in Y; then will SY be perpendicular to PT.

For since A is the middle point of NT (Art. 24, Cor. 1), therefore by parallels PN and AY (Art. 3), Y is the middle point of PT; and therefore the triangles SPY and STY, having their sides SP and PY equal to ST and TY respectively, and the side SY common, have their angles at Y equal.

Therefore SY is at right angles to PT, and conversely the foot of the focal perpendicular SY upon the tangent at P lies on the tangent at A.

(ii) Moreover, in the right-angled triangles SAY and SYP, since the tangents YA and YP to the parabola subtend equal angles at S (Art. 9),

$$SA : SY = SY : SP,$$

or

$$SY^2 = SA \cdot SP,$$

and SY^2 varies as SP, as was to be proved.

PROPOSITION IX.

27. *The exterior angle between any two tangents to a parabola is equal to the angle which either of them subtends at the focus.*

Let the tangents at P and Q intersect in T, and let them meet the axis in p and q. Take any point O in pS produced, and produce TS to any point t.

Then by Euclid I. 32, and because SPp is an isosceles triangle (Prop. VI.),

$$\angle PSO = SPp + SpP = 2SpP;$$

and similarly

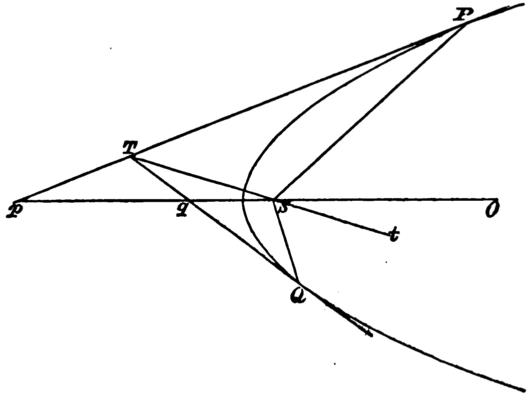
$$\angle QSO = 2SqQ.$$

Hence by addition (taking the case in which P and Q lie on opposite sides of the axis),

$$\angle PSQ = 2(SpP + SqQ) = 2(SpP + pqT),$$
$$= 2PTQ.$$

Therefore $\angle PTQ = \frac{1}{2}PSQ = PSt = QSt$; [Art. 9.

And therefore the exterior angle pTq between the tangents is equal to PST or QST, as was to be proved.



PROPOSITION X.

28. *The two tangents from any point to a parabola are the bases of a pair of similar triangles having a common vertex at the focus.*

With the construction of Art. 27 it has been shewn that

$$\angle PTQ = PSt.$$

The former angle is equal to $STP + STQ$, and the latter is equal to $STP + SPT$ (Euc. I. 32). Therefore, taking away STP from each,

$$\angle STQ = SPT.$$

And since the angles at S in the triangles STP and STQ are also equal (Art. 9), the third angle STP is equal to the third angle SQT, and the two triangles are similar, as was to be proved.

Hence $SP : ST = ST : SQ$,

or *the focal distance of the point of concourse of any two tangents to a parabola is a mean proportional to the focal distances of their points of contact.*

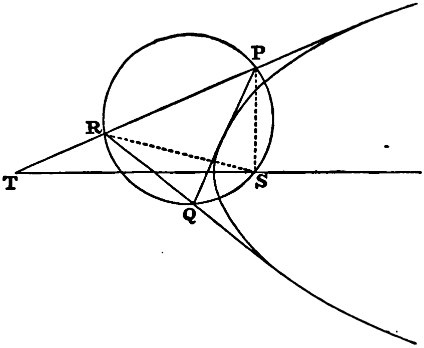
Corollary.

Since the angle STQ is equal to SPT, and therefore to SpP (Prop. VI.), the angle which either tangent makes with the axis (or with the diameter through T) is equal to the angle which the other makes with the focal distance of T.

PROPOSITION XI.

29. *The circumscribed circle of the triangle formed by any three tangents to a parabola passes through the focus.*

Let PQR be any triangle whose three sides touch a parabola, and let any one of its sides, as PR, meet the axis in T.



Then by the last corollary, considering the two tangents which meet in R,

$$\angle SRQ = STP;$$

and next considering the two tangents which meet in P,

$$\angle SPQ = STP.$$

Therefore the angles SPQ and SRQ are equal, and the points $PQRS$ lie on a circle; or in other words, the circle round PQR passes through the focus.

Corollary.

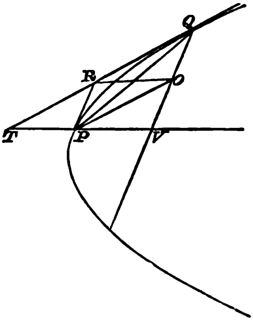
Four tangents to a parabola being given, the circumscribed circles of the four triangles which they determine meet in one point, viz. the focus of the parabola.

PROPOSITION XII.

30. *At any point of a parabola the subtangent to any diameter is double of the abscissa.*

Let Q be any point on a parabola, QV and PV its ordinate and abscissa to any diameter, and let the tangent at Q meet that diameter in T: we have to shew that TV is double of PV. [Def. Art. 17.

Let the tangents at P and Q meet in R, and let PO be drawn parallel to RQ to meet QV in O; then the tangent at P, being a vanishing ordinate to the diameter PV (Art. 13, Cor. 1), is parallel to QV; and therefore $PRQO$ is a parallelogram, and its diagonal RO bisects the diagonal PQ.



But the line drawn from R to bisect the chord of contact of the two tangents from R is a diameter of the parabola (Art. 13, Cor. 2): therefore RO is a diameter, and is parallel to the diameter PV. [Prop. II.

Hence by parallels,

$$PV = RO = PT,$$

or P bisects TV, and the subtangent TV is double of PV.

CHAPTER IV.

CENTRAL CONICS.

31. THE central conics are the ellipse and the hyperbola. It has been shewn (Art. 14) that these are also bifocal, and symmetrical with respect to both axes.

The *Abscissæ* or *Abscisses* of a point to any diameter of a central conic are the segments of the diameter made by the ordinate of the point. The *Central Abscissa*, which is also called absolutely the *Abscissa*, is the distance from the centre to the foot of the ordinate.

The principal axis of a central conic is distinguished as the *Transverse Axis*. The transverse and conjugate (Def. 4) axes of the ellipse are also called its *Major* and *Minor* axes. The term axis is sometimes used to denote the *finite* portion of either of these lines intercepted by the curve; but a special convention has to be made in case of the conjugate axis of the hyperbola, which does not meet the curve in real points. [Art. 33, Cor. 1.

The major and minor auxiliary circles are the circles described upon the transverse and conjugate axes as diameters, but when one *Auxiliary Circle* only is spoken of, the circle on the transverse axis is signified. It is easily seen from the next article that this is the limit to which the eccentric circle tends as its centre is made to approach the centre of the conic.

Supplemental Chords are chords drawn from any point on the curve to opposite extremities of any diameter.

The *Conjugate Parallelogram* is the figure formed by drawing parallels to each of two conjugate diameters through both extremities of the other.

32. *The segments of the transverse axis.*

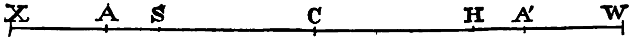
In any central conic the vertices A and A' divide SX so that

$$SA : AX = SA' : A'X;$$

therefore
$$SA : AX = SA \sim SA' : AX \sim A'X$$
$$= SA + SA' : AX + A'X,$$

or by Lemma B, if C be the centre of the conic,

$$SA : AX = 2CS : 2CA = 2CA : 2CX.$$



Thus each of the ratios $CS : CA$ and $CA : CX$ is equal to the eccentricity, and

$$CS . CX = CA^2.$$

The Ordinate.

PROPOSITION I.

33. *The square of the principal ordinate of any point on a central conic is in a constant ratio to the rectangle contained by its abscisses.*

Let A and A' be the vertices, PN the ordinate of any point P on the conic, Z and Z' the points in which AZ and $A'Z$ or their prolongations meet the directrix*.

If PM be a perpendicular to the directrix, and therefore parallel to the axis,

$$SP : SA' = PM : A'X \qquad \text{[Def. 1.}$$
$$= PZ' : A'Z';$$

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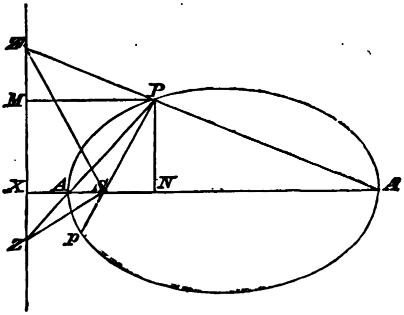
* For the hyperbola use the lithographed figure No. 2.

therefore SZ' bisects the angle PSX, and in like manner (if p be taken in PS produced) SZ bisects the supplementary angle pSX.



Hence the angle ZSZ' is a right angle, and

$$ZX.Z'X = SX^2 = \text{a constant.}$$

Moreover, since PN is parallel to the directrix,

$$PN : AN = ZX : AX,$$

and $$PN : A'N = Z'X : A'X.$$

Hence $$PN^2 : AN.A'N = ZX.Z'X : AX.A'X$$

$$= SX^2 : AX.A'X,$$

or PN^2 is in a constant ratio to $AN.A'N$.

Corollary 1.

Let the length CB be determined by the proportion,

$$CB^2 : CA^2 = SX^2 : AX.A'X.$$

Then in the ellipse

$$PN^2 : AN.A'N = PN^2 : CA^2 - CN^2 = CB^2 : CA^2,$$

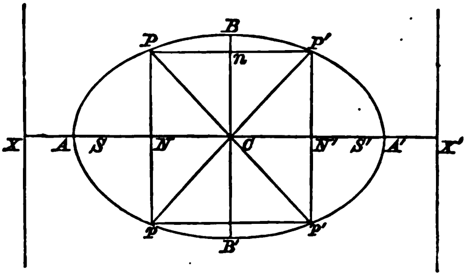
where CB is equal to the semi-axis conjugate, with which PN coincides when N is taken at C. In the hyperbola

$$PN^2 : AN.A'N = PN^2 : CN^2 - CA^2 = CB^2 : CA^2,$$

where we may agree to call *CB the length of the semi-axis conjugate,* although (Art. 2) this does not meet the curve.

Corollary 2.

If Pn be an ordinate to the conjugate axis, it is easily deduced that in the ellipse,



$$Pn^2 : CB^2 - Cn^2 = CA^2 : CB^2;$$

and in the hyperbola,

$$Pn^2 : CB^2 + Cn^2 = CA^2 : CB^2. \quad \text{[Fig. Art. 49.}$$

PROPOSITION II.

34. *The semi-axis conjugate is a mean proportional to the segments of the transverse axis made by either focus; and the latus rectum is a third proportional to the transverse and conjugate axes.*

Since $CS : CA = SA : AX = SA' : A'X,$ [Art. 32.
therefore

$$CS + CA : CA = SA + AX : AX = SX : AX$$

and $CS \sim CA : CA = SA' \sim A'X : A'X = SX : A'X.$

Hence $CS^2 \sim CA^2 : CA^2 = SX^2 : AX . A'X$

$$= CB^2 : CA^2, \quad \text{[Art. 33.}$$

or $$CS^2 \sim CA^2 = AS . A'S = CB^2.$$

If LSL' be the semi-latus rectum, so that AS, $A'S$ are the abscissæ of L,

$$SL^2 : AS.A'S = CB^2 : CA^2, \qquad \text{[Prop. I,}$$

and therefore $$SL^2 : CB^2 = CB^2 : CA^2.$$

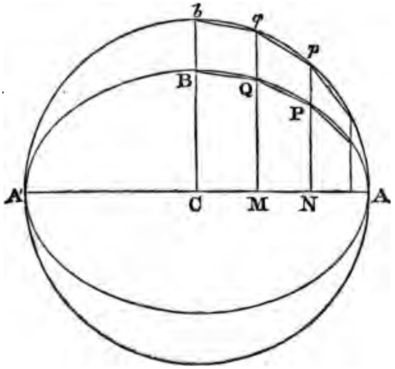
Therefore $SL.CA = CB^2$, and LL' is a third proportional to the transverse and conjugate axes.

PROPOSITION III.

35. *If the principal ordinates of all points on ellipse be produced outwards in the ratio of the major to the minor axis, the points to which they are produced lie on the circumference of the auxiliary circle, and conversely.*

Let P be any point on an ellipse, and let its principal ordinate PN be produced outwards to p in the ratio of CA to CB, so that

$$\begin{aligned} PN^2 : pN^2 &= CB^2 : CA^2 \\ &= PN^2 : AN.A'N. \qquad \text{[Prop. I.} \end{aligned}$$



Then, pN^2 being equal to $AN.A'N$, the locus of p is the circle on AA' as diameter, as was to be proved.

Conversely, if the ordinates pN, qM... in the circle be cut in the ratio of minority $CB : CA$, the points of section P, Q... will all lie on an ellipse whose semi-axes are equal to CA and CB.

Corollary.

Considering any two of the produced ordinates, the chords PQ and pq joining their extremities are easily seen to meet on the axis. Hence (making P, Q and p, q coalesce) we deduce that the tangents at P and p meet at a point T on the axis (Fig. Art. 45). Therefore the angle CpT is a right angle, and

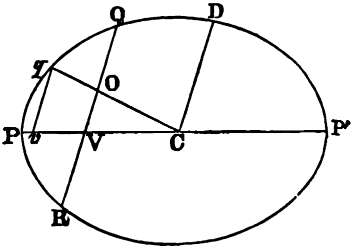
$$CN \cdot CT = Cp^2 = CA^2.$$

To draw the tangent at P, we have simply to draw the tangent pT to the circle at p, and join PT.

PROPOSITION IV.

36. *At any point on a central conic the square of the ordinate to any diameter is in a constant ratio to the product of the corresponding abscissæ.*

Let QVR be any double ordinate of a given diameter PP', and PV and $P'V$ the corresponding abscissæ. We have to shew that QV^2 (or RV^2) varies as $PV \cdot VP'$.



This follows at once from Art. 16, Cor. 1, where it is shewn that $QV \cdot VR$ (which is in this case equal to QV^2) is in a constant ratio to $PV.PV'$, so long as the directions of QR and PP' continue unchanged.

Put $CD^2 : CP^2$ equal to this constant ratio, so that

$$QV^2 : PV.VP' = CD^2 : CP^2.$$

In the ellipse it is evident that CD is equal to the semi-diameter to which QV is parallel. Also

$$PV.VP' = CP^2 - CV^2,$$

in all cases.

In the hyperbola also we may agree to call CD *the length of the semi-diameter conjugate to* CP, although it will be seen (in the section on conjugate diameters) that one and one only of every two conjugate diameters meets the curve in real points. Supposing CP to meet the hyperbola,

$$QV^2 : CV^2 - CP^2 = CD^2 : CP^2,$$

and

$$Qv^2 : Cv^2 + CD^2 = CP^2 : CD^2,$$

if Qv* be an ordinate to the diameter CD, which does *not* meet the curve.

Corollary.

Hence, and by Art. 16, Cor. 2, if FF' be the focal chord parallel to CD, and LL' be the latus rectum,

$$FF' : LL' = CD^2 : CB^2 = CD^2 : \tfrac{1}{2}LL'.CA. \quad \text{[Art. 34.}$$

Therefore

$$FF'.CA = 2CD^2,$$

or any focal chord is a third proportional to the transverse axis and the diameter parallel to the chord.

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* Draw Qv equal and parallel to VC in the lithographed figure No. 1, where the upper and lower branches do not belong to the hyperbola on which Q lies.

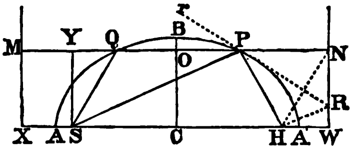
The Focal Distances.

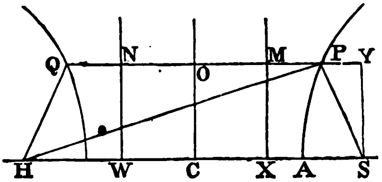
PROPOSITION V.

37. *The sum or difference of the focal distances of any point on a central conic is constant and equal to the transverse axis.*

Let H be the second focus (Art. 14) and W the foot of the corresponding directrix. Through any point P on the curve draw a parallel to the axis to meet the directrices in M and N. Then

$$SP : PM = HP : PN.$$





Therefore

$$SP \pm HP : PM \pm PN = SP : PM,$$

where $PM + PN$ is equal to NX in the ellipse, and $PM \sim PN$ is equal to NX in the hyperbola.

Therefore $SP \pm HP$ is constant, the upper sign being taken for the ellipse, and the lower sign for the hyperbola.

By making P coincide with A, it is easily seen that in either case the constant length is equal to the transverse axis, so that

$$SP \pm HP = AA'.$$

Corollary.

In the ellipse, if B be an extremity of the minor axis,

$$SB = HB = CA.$$

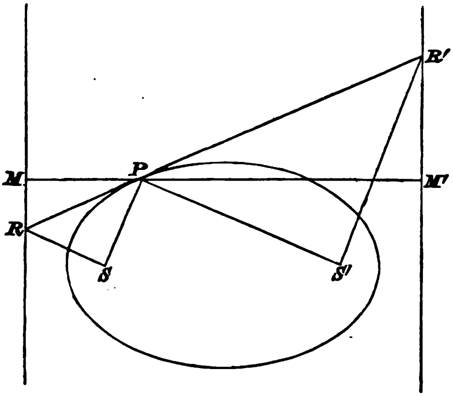
The Tangent.

PROPOSITION VI.

38. *The tangent at any point of a central conic makes equal angles with the focal distances of the point.*

Let the tangent at P meet the directrices in R and R', and let the parallel to the axis through P meet the directrices in M and M'. Then, if S and S' be the corresponding foci,

$$SP : S'P = PM : PM' = PR : PR',$$



or the sides about the angles at P in the triangles PSR and $PS'R'$ are proportionals.

Moreover the angles at S and S' in those triangles (being right angles) are equal.

Therefore the triangles are similar and their angles at P are equal (Euc. VI. 7); that is to say, the tangent at P makes equal angles with the focal distances SP and $S'P$, and conversely.

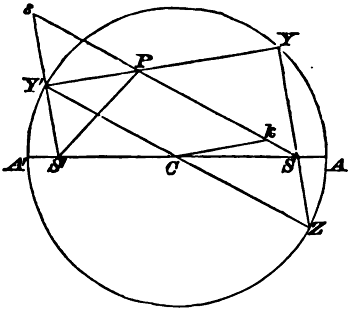
In the ellipse the tangent at P falls without the angle SPS', and bisects the angle which $S'P$ makes with SP produced (Fig. Art. 39). In the hyperbola (Fig. 4) the tangent bisects the angle SPS' internally.

PROPOSITION VII.

39. *The projections of the two foci upon any tangent to a central conic lie on the auxiliary circle.*

Let S and S' be the foci: Y and Y' the feet of the perpendiculars let fall from them upon the tangent at any point P. Then will Y and Y' lie on the auxiliary circle.

In the case of the ellipse, let SP produced meet $S'Y'$ in s; then the tangent PY' bisects the angle $S'Ps$ (Prop. VI.), and $Y's$ is equal to $Y'S'$.



And since Y' has been shewn to be the middle point of $S's$, and the centre C of the conic bisects SS', therefore CY' is parallel to Ss and equal to $\frac{1}{2}Ss$. That is to say,

$$CY' = \tfrac{1}{2}(SP + Ps) = \tfrac{1}{2}(SP + PS')$$
$$= CA, \qquad \text{[Prop. V.}$$

or Y' lies on the auxiliary circle. And in like manner it may be shewn that Y lies on the auxiliary circle.

In the case of the hyperbola, making the construction indicated in Figure 8, we may shew that

$$CY = \tfrac{1}{2}(S'P - SP) = CA,$$

and thus that Y and Y' lie on the auxiliary circle.

Corollary 1.

Conversely, if Y be any point on the auxiliary circle the straight line drawn from Y at right angles to SY will be a tangent to the conic. It is hence evident that the extremities of any focal chord YSZ of the auxiliary circle are the projections of the focus S upon a pair of *parallel tangents* to the conic.

Corollary 2.

If the diameter parallel to the tangent at P meet SP in k, then

$$Pk = CY' = CA.$$

PROPOSITION VIII.

40. *The rectangle contained by the perpendiculars from either focus of a central conic upon any two parallel tangents, or by the perpendiculars from the two foci upon any tangent, is constant and equal to the square of the semi-axis conjugate.*

Draw perpendiculars SY and SZ to any two parallel tangents. Then since Y and Z lie on the auxiliary circle (Prop. VII.) and YSZ is a straight line,

$$SY \, . \, SZ = SA \, . \, SA' = CB^2, \qquad \text{[Prop. II.}$$

as was to be proved.

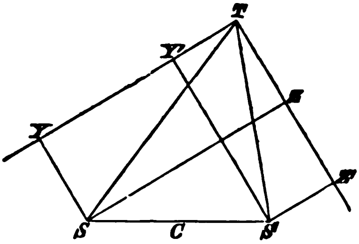
Again, with the construction of Art. 39, since $Y'YZ$ is a right angle $Y'Z$ is a diameter of the circle. Hence the triangles CSZ and $CS'Y'$, having their angles at C equal and their sides CS, CZ equal to CS', CY' each to each, are equal in all respects.

Therefore $\quad SY \, . \, S'Y' = SY \, . \, SZ = CB^2,$

as was to be proved.

Corollary.

If tangents be drawn to an ellipse from any point T, and if Y, Y' and Z, Z' be the projections of the foci upon them, then since $SY \, . \, S'Y' = SZ \, . \, S'Z'$, it is easily shewn by a



reductio ad absurdum that the angle STY is equal to $S'TZ$, or the two tangents are equally inclined to the focal distances of their point of concourse, each to each.

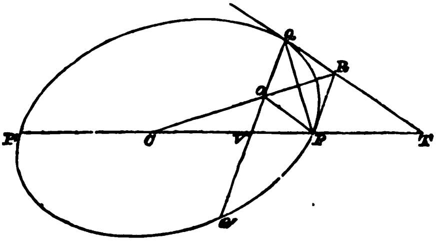
PROPOSITION IX.

41. *A variable tangent to a central conic meets any fixed diameter at a distance from the centre which varies inversely as the abscissa of its point of contact to that diameter.*

First Case[*].

Let PCP' be a given diameter which *meets* the curve in real points P and P'. Let the tangent at any point Q meet that diameter in T, and let QV be the ordinate of Q. Then we shall shew that

$$CV \, . \, CT = CP^2.$$



Let the tangent at P, which is parallel to QV, meet QT in R, and let PO be drawn parallel to RQ to meet QV in O. Then $OPRQ$ is a parallelogram, and its diagonal RO bisects PQ.

[*] Figure 5 (with suitable letters) may be used for the hyperbola.

But RO, since it bisects the chord of contact of the tangents from R, is a diameter and passes through the centre C of the conic. [Art. 13, Cor. 2.

Therefore by parallels,

$$CV : CP = CO : CR = CP : CT,$$

or $CV \,.\, CT$ is equal to CP^2, as was to be proved.

Second Case.

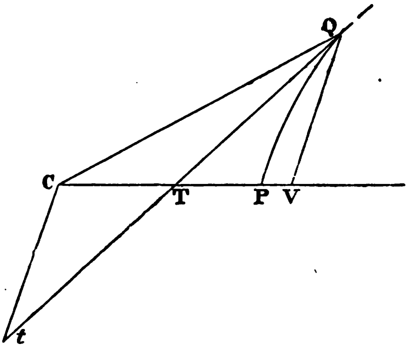
Let CP and CD be given conjugate semi-diameters of a hyperbola, whereof the former meets and the latter does *not meet* the curve. [Art. 36.

Draw QV and QT as in the first case, and produce QT to meet CD in t. Then by parallels,

$$Ct : CT = QV : VT,$$

or

$$\begin{aligned} QV \,.\, Ct : CV \,.\, CT &= QV^2 : CV \,.\, VT \\ &= QV^2 : CV^2 - CV \,.\, CT. \end{aligned}$$



Hence by the first case and by Art. 36,

$$\begin{aligned} QV \,.\, Ct : CP^2 &= QV^2 : CV^2 - CP^2 \\ &= CD^2 : CP^2, \end{aligned}$$

or

$$Cv \,.\, Ct = QV \,.\, Ct = CD^2,$$

if Cv be the abscissa of Q to the diameter CD.

Corollary.

If the tangent at any point P meet the transverse and conjugate axes in T and t, and if PN and Pn be ordinates to those axes, then

$$CN \, . \, CT = CA^2, \text{ and } Cn \, . \, Ct = CB^2. \qquad \text{[Fig. 6.}$$

The Normal.

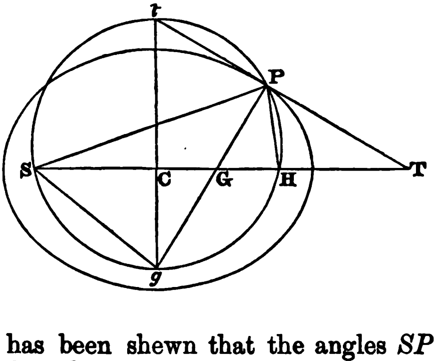
PROPOSITION X.

42. *The normal at any point bisects the angle between the focal distances of the point, internally in the case of the ellipse and externally in the case of the hyperbola.*

Let S and H be the foci, PG the normal at any point P, and TPt the tangent at that point.

Then in the ellipse, PG being at right angles to the tangent,

$$\angle \, SPG + SPt = HPG + HPT.$$



And it has been shewn that the angles SPt, HPT are equal and that the tangent falls without the angle SPH (Art. 38).

Therefore $$\angle SPG = HPG,$$
or the normal bisects SPH *internally.*

In the hyperbola it may be shewn in like manner that the normal at P bisects the angle SPH *externally.*

Corollary.

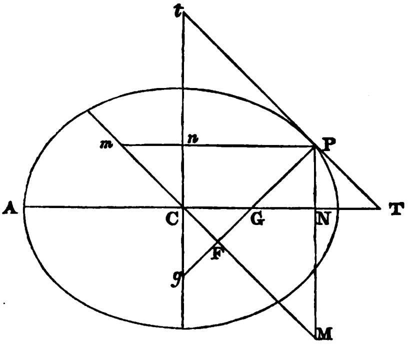
If the circle round SPH meet the conjugate axis in g and t, it is evident that the arcs Sg, Hg and the arcs St, Ht are equal. Hence Pg and Pt are the two bisectors of the angle SPH, and *the circle round SPH passes through the points in which the tangent and normal at P meet the conjugate axis.*

PROPOSITION XI.

43. *At any point of a central conic the normal, terminated by either axis, varies inversely as the central perpendicular upon the tangent or directly as the radius parallel to the tangent.*

(i) Let the tangent at any point P meet the transverse and conjugate axes in T and t, and let the normal meet them in G and g*.

Let PN and Pn be ordinates to those axes, and let PN and Pn or their prolongations meet the diameter parallel to



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* For the hyperbola use the lithographed figure No. 6.

the tangent at P in M and m, and let the normal meet that diameter in F.

Then, the angles at N and F being right angles, a circle goes round $FGNM$, and therefore

$$PG \,.\, PF = PN \,.\, PM = Cn \,.\, Ct$$
$$= CB^2. \qquad \text{[Art. 41, Cor.}$$

In like manner, the angles at n and F being right angles,

$$Pg \,.\, PF = Pn \,.\, Pm = CN \,.\, CT$$
$$= CA^2.$$

Therefore PG and Pg vary inversely as PF, or as the central perpendicular upon the tangent at P.

(ii) It will be shewn in Art. 45, Cor. 2 that

$$PG : CD = CD : Pg = CB : CA,$$

where CD is the semi-diameter conjugate to CP.

Corollary.

Hence

$$NG : CN = NG : Pn = PG : Pg = CB^2 : CA^2,$$

or *the subnormal varies as the abscissa.* [Def. Art. 17.

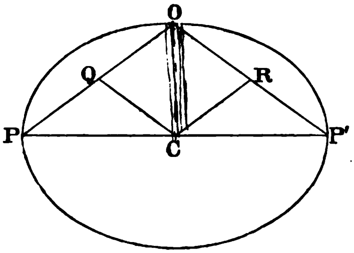
Conjugate Diameters.

PROPOSITION XII.

44. *Supplemental chords are parallel to conjugate diameters, and conversely.*

Let PCP' be any diameter and O any point on the curve. Then will the supplemental chords OP and OP' (Def. Art. 31) be parallel to a pair of conjugate diameters.

For if Q be the middle point of OP, and R the middle point of OP', the line CQ which bisects two sides of the triangle OPP' is parallel to the third side $P'O$, and in like manner CR is parallel to PO.



Therefore the diameters CQ and CR are conjugate, since each bisects one chord (and therefore all chords) parallel to the other. [Art. 13.

Conversely, from any given point O on the curve or from the extremities of any given diameter PP' there can be drawn a pair of supplemental chords OP and OP' parallel to any assumed pair of conjugate diameters.

In the hyperbola it is evident that of every two supplemental chords one lies within and the other without the curve, and hence that *one and one only of every two conjugate diameters meets the hyperbola.* We shall in consequence have occasion to give separate proofs of some of the properties of conjugate diameters for the special case of the hyperbola.

Corollary 1.

This proposition determines the relation between the directions of any two conjugate diameters. For in Art. 33, where AP and $A'P$ may be parallel to any two such diameters,

$$PN^2 : AN \,.\, A'N = CB^2 : CA^2;$$

and therefore if the ratio of PN to AN (or the direction of one of the diameters) be given, the ratio of PN to $A'N$ (or the direction of the conjugate diameter) is known. Conversely any two diameters whose directions are thus related will be conjugate provided that they lie in adjacent quadrants in the case of the ellipse (Fig. Art. 45), or in the same quadrant (Fig. 3) or opposite quadrants in the case of the hyperbola.

Corollary 2.

A pair of semi-diameters CP and CD (subject to the last-mentioned proviso) will therefore be conjugate if the ordinates PN and DR of their extremities be such that

$$PN : CR = CB : CA = DR : CN,$$

and conversely. To determine such a pair of semi-diameters in the ellipse, draw any pair of radii Cp and Cd *at right angles* in its auxiliary circle, draw the ordinates pN and dR of their extremities, and let these meet the ellipse in P and D; then it is evident that $pN = CR$ and $dR = CN$, and hence by Prop. III. that PN and DR are related as above. The same proportions hold in the case of the hyperbola; but it is to be noted that they implicitly assume the relation $CP^2 \sim CD^2 = CA^2 \sim CB^2$ (Art. 45), which must therefore be proved independently.

PROPOSITION XIII.

45. *The sum of the squares of any two conjugate diameters is constant in the ellipse, and the difference of their squares is constant in the hyperbola.*

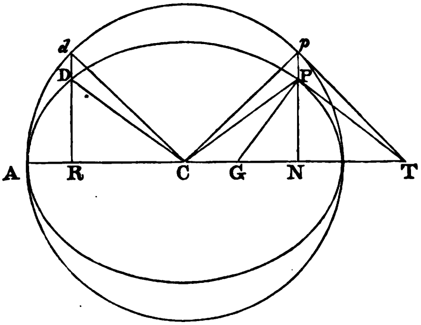
(i) Let CP and CD be conjugate semi-diameters of an ellipse, PN and DR the principal ordinates of their extremities, p and d the points in which those ordinates meet the auxiliary circle. Then pCd is a right angle, and pN is equal to CR, and dR to CN. [Art. 44, Cor. 2.

Therefore

$$\begin{aligned} pN^2 + dR^2 = pN^2 + CN^2 &= Cp^2 \\ &= CA^2, \end{aligned}$$

or (diminishing each of the lengths pN, dR, CA in the ratio of CB to CA)

$$PN^2 + DR^2 = CB^2. \qquad \text{[Prop. III.}$$



Moreover $$CN^2 + CR^2 = CN^2 + pN^2 \qquad \text{[Art. 44, Cor. 2.}$$
$$= CA^2.$$

By addition $$CP^2 + CD^2 = CA^2 + CB^2, \qquad \text{[Euc. I. 47.}$$
as was to be proved.

(ii) In the hyperbola, as will be proved in Art. 52,
$$CP^2 \sim CD^2 = CA^2 \sim CB^2.$$

Corollary 1.

If P be any point on an ellipse, $(SP + S'P)^2 = 4CA^2$, and $SP^2 + S'P^2 = 2CS^2 + 2\,CP^2$ (Lemma D). Therefore, subtracting and dividing by two,

$$SP \,.\, S'P = 2CA^2 - CS^2 - CP^2 = CA^2 + CB^2 - CP^2$$
$$= CD^2,$$

as may also be proved in like manner for the hyperbola.

Corollary 2.

If the normal at P meet the transverse axis in G, then by similar triangles (CD being at right angles to the normal)

$$PG : CD = PN : CR = CB : CA.$$

And in like manner it may be shewn that

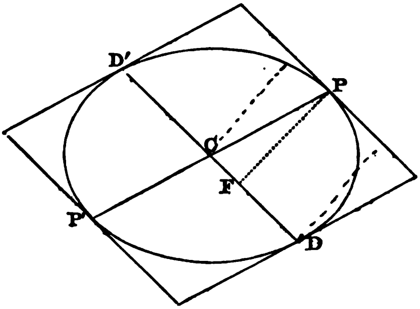
$$Pg : CD = CA : CB,$$

where Pg is the normal terminated by the conjugate axis.

PROPOSITION XIV.

46. *The conjugate parallelogram is of constant area and equal to the rectangle contained by the axes.*

Let PCP' and DCD' be a pair of conjugate diameters, and let a conjugate parallelogram be constructed by drawing parallels to each of them through the extremities of the other. [Def. Art. 31.



Let the normal at P meet DD' in F and the transverse axis in G. Then since

$$PG : CD = CB : CA, \quad \text{[Art. 45, Cor. 2.}$$

therefore

$$PF.PG : PF.CD = CB^2 : CA.CB.$$

Hence, the antecedents being equal by Prop. XI.,

$$PF.CD = CA.CB;$$

and the conjugate parallelogram is equal to $4PF.CD$ or $AA'.BB'$[*].

---

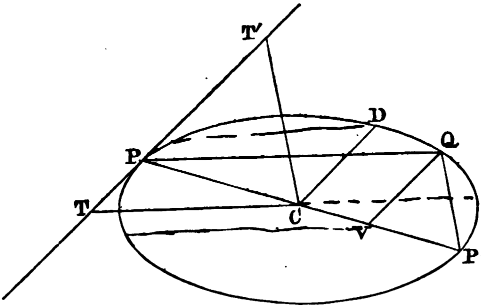
[*] A second proof for the hyperbola alone will be found in Art. 52 (ii).

In the ellipse the parallelogram described as above completely envelopes the curve: in the hyperbola two of its sides only touch the curve (Fig. Art. 50).

PROPOSITION XV.

47. *The intercepts on any tangent between the curve and any two conjugate diameters contain a rectangle equal to the square of the semi-diameter parallel to the tangent.*

Let the tangent at P meet any two conjugate diameters in T and T'. Draw the diameter PCP' and the conjugate semi-diameter CD, and draw the supplemental chords QP and $P'Q$ parallel to CT and CT'. [Art. 44.



Then by similar triangles, if QV be an ordinate to PP',

$$PT : CP = QV : PV,$$

and

$$PT' : CP = QV : P'V.$$

Hence

$$PT.PT' : CP^2 = QV^2 : PV.P'V$$
$$= CD^2 : CP^2, \quad \text{[Art. 36.}$$

or $PT.PT'$ is equal to CD^2, as was to be proved.

Corollary.

In the case of the hyperbola the points T and t in which any two conjugate diameters meet the tangent at Q lie on the same side of Q, as in the second figure of Art. 41. If the two conjugate diameters be supposed to coalesce, the intercept QT becomes equal to the parallel semi-diameter CD, as will be noticed more particularly in the next chapter.

CHAPTER V.

THE ASYMPTOTES.

48. IF a straight line and a curve, being produced, continually approach one another but never actually meet until they are produced to infinity, the straight line is said to be an *Asymptote* of the curve. It will be seen that every hyperbola has two asymptotes.

If CE and CP be two diameters of a hyperbola lying in the same quadrant, and EPN be an ordinate to the transverse axis, the diameters will be conjugate provided that

$$EN . PN : CN^2 = CB^2 : CA^2. \qquad \text{[Art. 44, Cor. 1.}$$

If CP be made to coalesce with CE, it follows that

$$EN : CN = CB : CA,$$

a relation which determines the position of the *self-conjugate* diameter CE. The curve has also a second self-conjugate diameter CE' making the same angle as CE with the axis.

Since one of every two conjugate diameters meets and the other does not meet the hyperbola (Art. 44), a diameter which is self-conjugate should meet and yet not meet the curve: accordingly it will appear that the self-conjugate diameters coincide with the asymptotes, which meet the curve at infinity and do not meet it at any assignable distance from the centre.

For the sake of uniformity of enunciation we shall at the outset speak of the lines CE and CE' as the asymptotes, and shall then shew conversely that they possess the property from which the term asymptote is derived. [Prop. I. Cor.

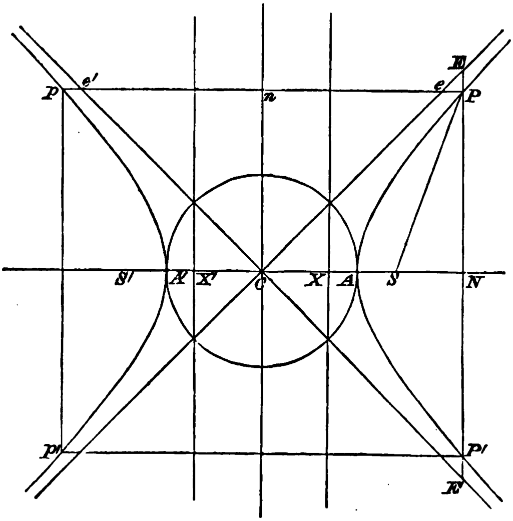
It is to be noticed that the asymptotes are the diagonals of the rectangle formed by drawing parallels to each axis through both extremities of the other. On the tangent at A take a length AL equal to CB, then will CL be equal to CS (Art. 34), and $CL : CA$ to the eccentricity. Also it may be easily proved that the feet of the focal perpendiculars upon the asymptotes lie on the auxiliary circle, and that each perpendicular is equal to CB.

PROPOSITION I.

49. *The rectangle contained by the distances of any point on a hyperbola from its two asymptotes is of constant magnitude.*

(i) If the principal ordinate of any point P on the hyperbola meet the axis in N and the asymptotes in E and E', then

$$PN^2 + CB^2 : CN^2 = CB^2 : CA^2 \qquad \text{[Art. 33.}$$
$$= EN^2 : CN^2; \qquad \text{[Art. 48.}$$



and therefore $PN^2 + CB^2$ is equal to EN^2 and

$$PE \,.\, PE' = EN^2 - PN^2 = CB^2.$$

(ii) Next let Pe be drawn in any specified direction to meet CE. Then the triangle PEe is given in species and Pe varies as PE, and therefore $Pe\,.\,PE'$ is constant.

In like manner, if Pe' be drawn in any specified direction* to meet CE', the length Pe' varies as PE', and therefore $Pe\,.\,Pe'$ is constant.

Taking for example the case in which PO the distance of P from one asymptote is measured parallel to the other (Fig. Art. 52), and CO is therefore equal to the distance of P from the latter asymptote measured parallel to the former, we have $PO\,.\,CO$ constant, and it may be shewn by making P coincide with the vertex that

$$PO\,.\,CO = \tfrac{1}{4}(CA^2 + CB^2) = \tfrac{1}{4}CS^2. \qquad \text{[Art. 48.}$$

Corollary.

Since $PE\,.\,PE'$ is constant and PE' continually increases with CN, therefore PE at the same time continually decreases. Hence CE and the curve continually approach one another but never actually meet until produced indefinitely.

PROPOSITION II.

50. *The intercepts on any tangent to a hyperbola between the curve and its asymptotes are equal to one another and to the parallel semi-diameter; and the opposite intercepts on any chord between the curve and its asymptotes are equal to one another.*

(i) Let the tangent at P be parallel to the semi-diameter CD, and let it meet any two conjugate diameters in T and T'', so that

$$PT\,.\,PT'' = CD^2. \qquad \text{[Art. 47.}$$

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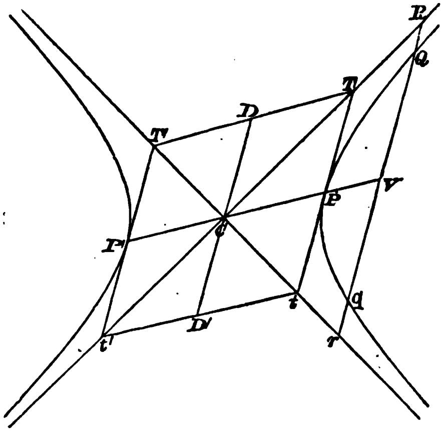
* In this construction Pe and Pe' are not required to be in the same straight line. In the special case in which Pee' is a straight line and parallel to the axis it may be shewn that $Pe\,.\,Pe' = CA^2$. The same may be shewn (Art. 51) if CA be *any* radius, regarded as fixed for the time being.

Let CT' coalesce with CT, so that CT becomes a self-conjugate diameter or asymptote; then PT^2 is equal to CD^2,



and in like manner, if the same tangent meet the other asymptote in t, Pt^2 is also equal to CD^2. Therefore

$$PT = CD = Pt,$$

as was to be proved.

It readily follows that every two parallel tangents as TPt and $T'P't'$ terminated by the asymptotes are sides of a conjugate parallelogram (Def. Art. 31), and conversely that every conjugate parallelogram has its diagonals coincident with the asymptotes.

(ii) Next let any chord Qq parallel to the tangent TPt be produced to meet the asymptotes in R and r. Then the diameter CP bisects Rr, and it also bisects the chord Qq. [Art. 13, Cor. 1.

Therefore the opposite intercepts QR and qr are equal and the opposite intercepts Qr and qR are equal, as was to be proved. And the same may be shewn in like manner if the chord be supposed parallel to the diameter PP' which meets the curve in real points.

PROPOSITION III.

51. *Any chord of the asymptotes is divided at either of the points in which it meets the curve into segments to which the parallel radius is a mean proportional.*

In the figure of Prop. II. let V be the middle point of Qq and Rr. Then by Art. 36 and by parallels (supposing for example that Q and q lie on the same branch of the curve),

$$QV^2+CD^2 : CV^2 = CD^2 : CP^2 = PT^2 : CP^2$$
$$= RV^2 : CV^2.$$

Therefore, the antecedents being equal,

$$RQ \,.\, Qr = Rq \,.\, qr = RV^2 - QV^2 = CD^2,$$

or CD is a mean proportional to RQ, Qr and to Rq, qr*.

PROPOSITION IV.

52. *The difference of the squares of any two conjugate semi-diameters of a hyperbola is equal to the difference of the squares of the semi-axes; and the triangle contained by the asymptotes and any tangent is equal to the rectangle contained by the semi-axes.*

(i) Let the tangent at P meet the asymptotes in L and M. Draw PY perpendicular to CL, and bisect CL in O. Then, P being the middle point of LM (Prop. II.), OP is parallel to the asymptote CM, and the triangle OYP is given in species and OY varies as OP.

Also, by Euc. I. 47 and Lemma A,

$$CP^2 \sim PL^2 = CY^2 \sim LY^2 = 4CO \,.\, OY,$$

where OY varies as OP, and therefore inversely as CO.

[Prop. I.

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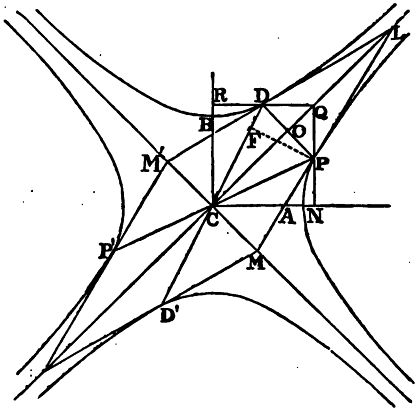
* We may deduce the same from Prop. I. by taking any chord parallel to a fixed diameter, and then supposing it to coalesce with the diameter itself (if it meet the curve), or with the parallel tangent.

Therefore $CP^2 \sim CD^2$ or $CP^2 \sim PL^2$ has a constant magnitude, which may be shewn by taking P at the vertex to be equal to $CA^2 \sim CB^2$.



(ii) The rectangle $CL \, . \, CM'$ is equal to $2CO \, . \, 2PO$ or CS^2 (Art. 49). Therefore the triangle LCM is of constant area, and it may be shewn by taking P at the vertex to be equal to $CA \, . \, CB$.

The *Conjugate Parallelogram*, which is four times the triangle CTt (Art. 50), is therefore equal to $AA' \, . \, BB'$.

CHAPTER VI.

THE EQUILATERAL HYPERBOLA.

53. THE *Equilateral Hyperbola* is a hyperbola whose latus rectum is equal to its transverse axis. Its two axes being equal (Art. 34), its asymptotes are at right angles, and it is therefore called also the *Rectangular Hyperbola.* Its eccentricity is the ratio of the diagonal to the side of a square, the foot X of the S-directrix bisects CS, and

$$\tfrac{1}{2}\,CS^2 = CA^2 = 2CX^2 = 2SX^2. \qquad \text{[Art. 32.}$$

Also $$PN^2 = AN\,.\,A'N = CN^2 - CA^2, \qquad \text{[Art. 33.}$$

and $$QV^2 = PV\,.\,P'V = CV^2 - CP^2 \text{*}, \qquad \text{[Art. 36.}$$

since by Prop. II. every diameter is equal to its conjugate.

The normal at P terminated by either axis is equal to CP or CD (Art. 45, Cor. 2), and therefore also to the intercept on any tangent between the curve and either asymptote. [Art. 50.

It remains to prove certain of the more distinctive properties of this variety of the hyperbola, which bears the same kind of relation to the general hyperbola that the circle (or equilateral ellipse) bears to the general ellipse.

PROPOSITION I.

54. *The angles between any two diameters of an equilateral hyperbola are equal to the angles between the conjugate diameters.*

Let any two diameters meet the S-directrix in V and V'; and draw SZ at right angles to SV and SZ' at right angles to SV', so that SZ and SZ' are the directions of the diameters conjugate to CV and CV'. [Art. 13.

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* This may also be deduced as a corollary from Prop. I.

Then since X is the middle point of CS, [Art. 53.

$$\angle VCV' = VSV' = ZSZ',$$

or the diameters CV and CV' are inclined at the same angles as their conjugates. Thus *any two conjugate diameters* (Fig. Art. 55) make equal angles with the two axes: they therefore *make equal angles with either asymptote, and make complementary angles with either axis.*

Corollary 1.

Let PP' be any diameter, Q and R any two points on the curve; then since supplemental chords are parallel to conjugate diameters (Art. 44) the angle QPR is equal or supplementary to $QP'R$, and therefore *any chord of an equilateral hyperbola subtends equal or supplementary angles at the extremities of any diameter.*

Corollary 2.

In the second figure of Art. 41 (supposing the hyperbola to be equilateral) it may be shewn by similar triangles that

$$QT \,.\, Qt = CQ^2;$$

and likewise that $$CV \,.\, VT = QV^2,$$

or $$CV \,.\, CT = CV^2 - QV^2 = CP^2.$$

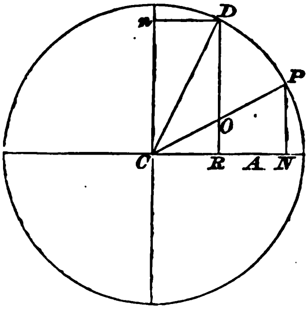
PROPOSITION II.

55. *Conjugate diameters and diameters at right angles are equal in the equilateral hyperbola; and the ordinates and abscissæ of the extremities of any two conjugate semi-diameters are alternately equal to one another.*

(i) The difference of the squares of any two conjugate diameters being equal to the difference of the squares of the axes (Art. 45), and the axes being in this case equal, it follows that every diameter is equal to its conjugate.

(ii) Let CP and CD be conjugate semi-diameters, PN and DR ordinates to the axis; and let PN produced meet the hyperbola again in P', so that $CP' = CP = CD$. Then

$$\angle P'CD = PCN + DCN = \text{a right angle}, \qquad \text{[Prop. I.}$$



or any two equal semi-diameters CP' and CD lying in adjacent quadrants are at right angles, and conversely.

(iii) Since the triangles CPN and CRD are similar (Prop. I.), and $CP = CD$, it readily follows that

$$PN = CR, \text{ and } DR = CN,$$

as was to be proved.

Corollary.

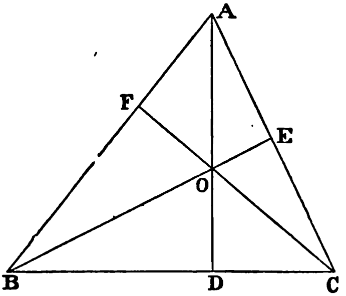
Hence and from the relation $CN^2 - PN^2 = CA^2$ it may be deduced that $\triangle CPD = \frac{1}{2} CA^2$. The conjugate parallelogram is therefore equal to $4\, CA^2$.

PROPOSITION III.

56. *If an equilateral hyperbola circumscribes a triangle it passes through its orthocentre, and conversely.*

Let AD, BE, CF be the three perpendiculars of a triangle ABC, and O the point in which they cointersect, which is called the orthocentre of the triangle.

Let a hyperbola be supposed to pass through the four points $ABCO$. Then since $AD.DO = BD.DC$, its diameters parallel to AD and BC are equal (Art. 16, Cor. 2), and in like manner its diameters parallel to BE and AC are equal, as also are those parallel to CF and AB.



Hence the hyperbola through $ABCO$, having more than one pair of equal diameters at right angles, must be equilateral (Prop. II.); and it may be inferred conversely that every equilateral hyperbola circumscribing a triangle ABC must pass through its orthocentre O.

CHAPTER VII.

THE CONE.

57. AN unlimited straight line which passes through a fixed point in space and makes a constant angle with a fixed straight line through the point generates a surface which is called a *Cone*. The fixed point is called the *Vertex*, the fixed line the *Axis*, and the variable line in any position is called a *Side* or a *Generating Line* of the cone. The constant angle between any two opposite sides of the cone is called its *Vertical Angle*. The complete cone consists of two infinite sheets situated on opposite sides of the vertex, as in the last figure of this chapter.

This species of cone is more fully described as the *right circular* cone, the sections of it made by planes at right angles to its axis being evidently circles. Any such section may be regarded as the *Base* of the cone.

In the special case in which the vertex is at infinity, and the generating lines are therefore all parallel to the axis and at right angles to the base, the surface is called no longer a cone but a *Cylinder*.

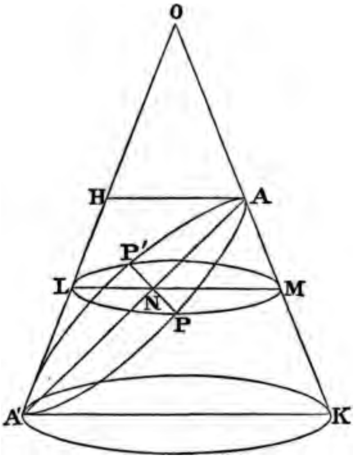
In what follows we shall consider a plane through the axis to be the *Plane of Reference* and the *Sections* to be made by planes at right angles thereto, so that the principal axis of a section will always lie in the plane of reference. It will appear from the following propositions that a plane section of a cone is in general a parabola, an ellipse or a hyperbola.

The Ordinate.

PROPOSITION I.

58. *The square of the principal ordinate in any section varies as the rectangle contained by the corresponding abscissæ.*

Let AA' be the axis of the section and PN the perpendicular upon it from any point P of the section. Draw the circular sections of the cone through P, A, A', and let their diameters in the plane of reference be LM, AH, $A'K$ respectively.



Then by similar triangles

$$LN : A'N = AH : AA',$$

and $$MN : AN = A'K : AA';$$

and by a property of the circle

$$PN^2 = LN \,.\, MN.$$

Therefore $PN^2 : AN \,.\, A'N = AH \,.\, A'K : AA'^2$,

or the locus of P is an ellipse or a hyperbola, according as the plane of the section cuts all the generating lines of the cone on the same side of the vertex, as above, or cuts both sheets of the cone, as in the last figure of Art. 59. In either case the conjugate axis is a mean proportional to AH and $A'K$, and therefore *the semi-axis conjugate is a mean proportional to the perpendiculars from the vertices of the section to the axis of the cone.*

When the plane of the section is parallel to a side of the cone, it may be shewn in like manner that PN^2 varies as AN and the section is a parabola.

Corollary.

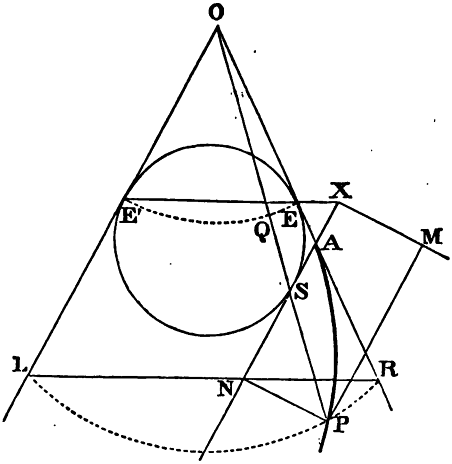
In the cylinder (in which all circular sections are equal) an oblique section is always an ellipse having its minor axis equal to the diameter of the base, whilst its major axis may be of any length greater than that diameter. Conversely *any ellipse may be regarded as a plane section of a right cylinder described on a circular base equal to the minor auxiliary circle of the ellipse.*

The Focal Spheres.

PROPOSITION II.

59. *If a sphere be inscribed in a cone so as to touch the plane of a section, the point of contact of the sphere with the plane will be a focus of the section, and the plane of contact of the sphere with the cone will meet the plane of the section in the corresponding directrix.*

(i) Let a sphere be drawn touching a cone along the circle EQE', and touching the plane of a section at the point S; then will S be a focus of the section, and the corresponding directrix will be the line MX in which the plane of contact EQE' meets the plane of the section.



For let P be any point on the section, PY* a perpendicular to the plane of contact, PM a perpendicular to MX, and Q the point in which the side of the cone through P meets the plane of contact.

Then since the angle QPY is always equal to half the vertical angle of the cone, and the angle MPY to the angle between the axes of the section and the cone, it follows that $PQ : PY$ and $PY : PM$ and therefore also $PQ : PM$ are constant ratios.

Hence, the tangents PS and PQ to the sphere being equal, SP also is in a constant ratio to PM; that is to say, the locus of P is a conic having the point S and the line MX for a focus and directrix.

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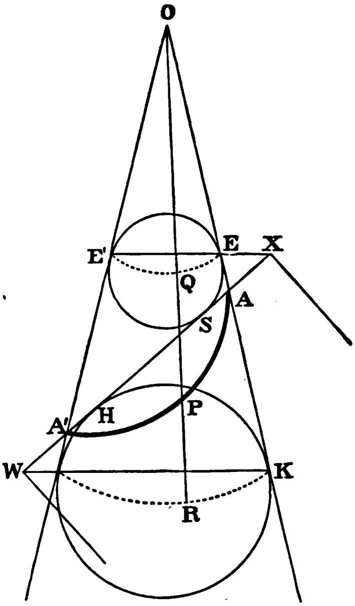
* This line is to be supplied in the diagram.

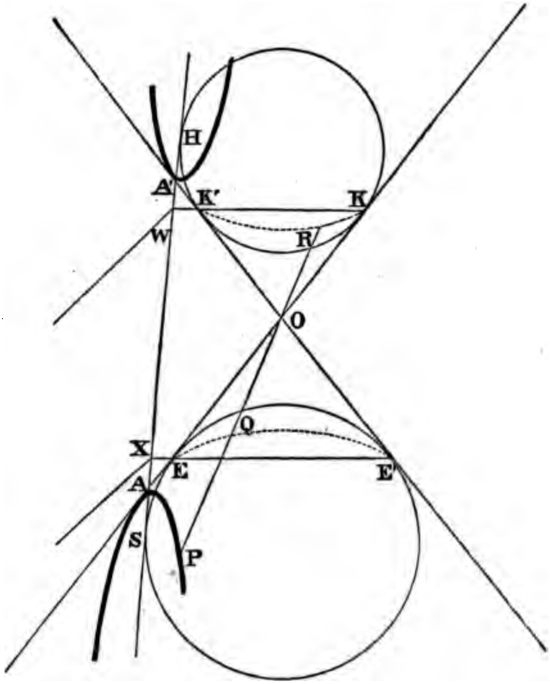
(ii) Every elliptic section has two focal spheres touching its plane on opposite sides: thus the two foci S, H and their directrices are determined.



In the annexed figure it may be shewn (antecedently to the determination of the directrices) that

$$SP + HP = QR = AA'.$$

(iii) Every hyperbolic section has two focal spheres touching its plane on the same side, as in the annexed figure.



In this case it may be shewn in like manner that

$$SP \sim HP = PR = AA'.$$

CHAPTER VIII.

ORTHOGONAL PROJECTION.

60. THE *Orthogonal Projection*, or briefly the *Projection* of any point in space upon a plane, is the foot of the perpendicular let fall from the point to the plane. The projection of any line or figure is determined by the projections of its several points. It is evident that the projections of any figure upon parallel planes are equal and similar.

PROPOSITION I.

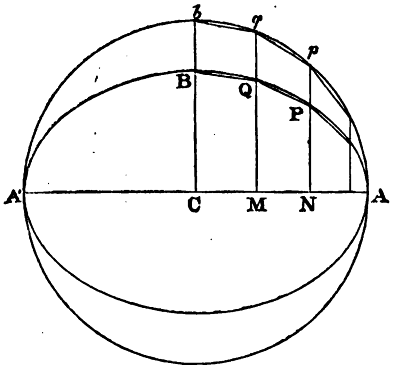
61. *Any ellipse may be projected into a circle equal to its minor auxiliary circle; and a circle may be projected into an ellipse of any eccentricity having its major axis equal to the diameter of the circle.*

(i) Describe a right cylinder upon a circular base equal to the minor auxiliary circle of the given ellipse; then the ellipse may be placed so as to coincide with one of the plane sections of the cylinder. [Art. 58, Cor.

The plane of the section is to be taken at such an inclination to the plane of the base that the projection of the major axis of the section upon the base may be equal to its minor axis.

(ii) All the ordinates to a diameter AA' in a circle are cut in the same ratio of minority $CB : CA$ by an ellipse whose major axis is AA' and whose minor axis is equal to $2CB$. [Art. 35.

Hence if the plane of the circle be turned about AA' through a certain angle[*], every point P on the ellipse will



lie vertically under the corresponding point p of the circle, or the ellipse will be the orthogonal projection of the circle. Thus the circle is projected into an ellipse having its major axis equal to AA', whilst its eccentricity increases with the acute angle between its plane and the plane of the circle and may be of any magnitude between zero and unity.

PROPOSITION II.

62. *The projections of parallel straight lines are parallel straight lines, and every line or segment in a system of parallels is in the same ratio to its projection.*

(i) The projection of a straight line upon a plane is the common section of that plane with the plane drawn at right angles to it through the line, since their common section evidently contains the projections of all points upon the line and of those only.

Let Mq' be a finite straight line and $M'Q'$ its projection, so that the plane $Mq'Q'M'$ is at right angles to the plane of projection.

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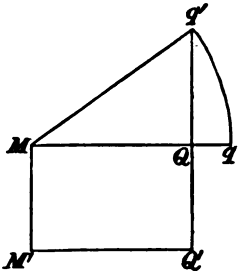
[*] In both cases of the proposition the cosine of the angle between the two planes must be equal to $CB : CA$.

Then if Mq' be regarded as a variable line belonging to a system of ***parallels***, the planes by which it is projected



will be parallel, and their common sections $M'Q'$ with the plane of projection will be parallel to one another, as was to be proved.

(ii) Moreover, if MQ be drawn equal and parallel to $M'Q'$, the angle $q'MQ$ will be constant, and the ratio of Mq' to MQ or $M'Q'$ will therefore be invariable, as was to be proved.

In the special case in which the original parallels and their projections are at right angles to the common section of their planes the constant angle $q'MQ$ is equal to *the angle between the planes.*

PROPOSITION III.

63. *Any area in one plane is to its projection upon any other plane in a ratio which depends only upon the angle between the planes.*

In the one plane draw any number of perpendiculars pN, qM,... to the common section of the planes, and let PN, QM,... be their projections upon the other plane, so that

$$PN : pN = QM : qM = CB : CA,$$

where $CB : CA$ is a ratio determined by *the angle between the planes**. [Prop. II.

---

* Compare the figure of Art. 61, supposing the planes of the two curves to be inclined at an angle whose cosine is equal to $CB : CA$.

It follows that every rectilinear figure $pNMq$ determined by the pair of perpendiculars pN, qM is to its projection $PNMQ$ as CA to CB; and the aggregate of any number of such figures is in the same ratio to the aggregate of their projections.

But any rectilinear figure in the primitive plane may be divided into elements by means of perpendiculars drawn as above; and any curvilinear figure may be regarded as the limit of a rectilinear figure whose adjacent angular points are indefinitely near to one another. Therefore any area whatever in the one plane is to its projection upon the other plane in the ratio of CA to CB, as was to be proved.

PROPOSITION IV.

64. *The points of concourse of lines and of their projections correspond to one another, and the tangent to a curve at any point corresponds to the tangent at the projection of the point.*

(i) Since the projection of any line is determined by the projections of its several points, if any number of lines straight or curved pass through a point, their projections must all pass through the projection of the point. For example, if a chord of any curve be drawn through a *fixed* point its projection will pass through the corresponding fixed point in the plane of projection.

(ii) If a straight line and a curve intersect in adjacent points P and Q, the projections of the straight line and the curve will intersect at the projections p and q of those points. Hence, the projecting lines Pp and Qq being always parallel, if the points P and Q coalesce their projections must also coalesce; that is to say, the tangent to the original curve at P projects into the tangent at p to the projection of the curve.

We shall conclude by briefly indicating the method of applying these propositions, with reference in the first instance to the ellipse.

65. *The Area of the Ellipse.*

It may be shewn by projecting a circle into an ellipse that the area of any ellipse is to that of its auxiliary circle as CB to CA, and hence that the area of the ellipse is equal to $\pi . CA . CB$.

66. *Diameters.*

The middle points of any system of parallel chords in a circle may be projected into the middle points of a system of parallel chords in an ellipse. But in the circle parallel chords are bisected by a straight line; therefore in the ellipse also parallel chords have their middle points in a straight line. Hence it appears that diameters at right angles in the circle correspond projectively to conjugate diameters in an ellipse.

67. *The Segments of Chords.*

Project an ellipse into a circle, or a circle into an ellipse. Let POQ, $P'OQ'$ be any two intersecting chords of the ellipse and CD, CD' the parallel semi-diameters: poq, $p'oq'$ the corresponding chords of the circle and cd, cd' the parallel radii.

Then by Prop. II. and by known properties of the circle,

$$\begin{aligned} OP . OQ : CD^2 &= op . oq : cd^2 \\ &= op' . oq' : cd'^2 \\ &= OP' . OQ' : CD'^2, \end{aligned}$$

or the rectangles contained by any two intersecting chords of an ellipse are as the squares of the parallel semi-diameters.

68. *The Tangent.*

Any two tangents to a circle meet upon the diameter bisecting their chord of contact, viz. at a point T such that

$$CV . CT = CP^2,$$

where V is the middle point of the chord, and P an extremity of the diameter CT. It readily follows by orthogonal projection that the same is true in the ellipse; and in like manner the property of Art. 47 may be first proved for the circle and then transferred by projection to the ellipse.

69. *Properties of Polars.*

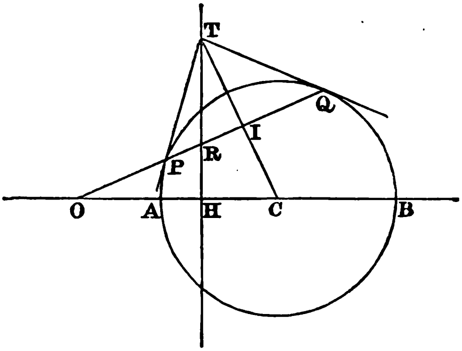
(*a*) *If a chord of a circle passes through a fixed point the tangents at its extremities intersect on a fixed straight line, and conversely.*

In a circle of radius CA draw any chord PQ through a point O, and let TP and TQ be the tangents at its extremities. Draw the perpendicular TH to CO, and let TH meet the chord in R, and let CT meet it in I.

Then since the angles at H and I are right angles the points H, I, T, O are concyclic, and

$$CH.CO = CI.CT = CA^2.$$

Hence if O be a fixed point, CH is constant and RH is a fixed straight line, and conversely.



The line RH is called the *Polar* of O, and the point O is called the *Pole* of RH.

If the point O be taken without the circle its polar will be the chord of contact of the tangents from O: if the point be taken within the circle its polar will lie wholly without the curve.

(*b*) *Any chord of a circle is divided harmonically by any point through which it passes and the polar of the point.*

For in the preceding figure the points C, I, R, H are concyclic, so that

$$\begin{aligned} OR \cdot OI = OH \cdot OC &= CO^2 \sim CO \cdot CH \\ &= CO^2 \sim CA^2 \\ &= OP \cdot OQ. \end{aligned}$$

Hence by Lemma B, since I is the middle point of PQ,

$$2OP \cdot OQ = 2OR \cdot OI = OR\,(OP + OQ),$$

and therefore PQ is divided harmonically at O and R.

These properties of polars may be extended to the ellipse by orthogonal projection. We may remark in passing that they are also true of the general conic: thus (to take a special case) the directrix is the polar of the focus. [Art. 7, Cor.

70. *The Equilateral Hyperbola.*

It readily follows from the property of the principal ordinate (Art. 33) that an equilateral hyperbola may be projected into a hyperbola of any eccentricity, and *vice versa*.

In the equilateral hyperbola let the length of any semi-diameter CD which does not meet the curve be *defined* by the condition that it shall be equal to the conjugate semi-diameter CP: let it be granted further that

$$PN^2 \sim DR^2 = CN^2 \sim CR^2 = CA^2, \qquad \text{[Art. 55.}$$

and that the triangle CPD is equal to CA^2.

It may then be deduced by projection that in the general hyperbola the difference of the squares of any two conjugate semi-diameters is equal to $CA^2 \sim CB^2$, and the area of the conjugate parallelogram to $AA' \cdot BB'$.

In like manner the property of the ordinate to any diameter in the general hyperbola (Art. 36) may be deduced from the special case of the equilateral hyperbola.

EXAMPLES.

The General Conic.

1. The distance of any point outside a conic from the focus is to its perpendicular distance from the directrix in a ratio greater than the eccentricity, and conversely.

2. If an ellipse, a parabola, and a hyperbola have the same focus and directrix, the ellipse lies wholly within the parabola, and the parabola wholly within the hyperbola.

3. Conics having the same focus and directrix do not meet.

4. If SL be the semi-latus rectum, and SD be drawn parallel to PR (Art. 4) to meet the directrix, then

$$SP : PR = SL : SD.$$

5. The segments of any focal chord subtend equal (or supplementary) angles at the foot of the directrix.

6. If chords PR, QR be produced to meet the directrix in p, q, the angle between the focal radii to p, q will be equal to half the angle between the focal radii to P, Q.

7. If the focus and two points of a conic be given, the directrix must pass through one of two fixed points.

8. Having given two points on a conic and its focus and eccentricity, shew how to describe the curve.

9. Having given two points on a conic and its directrix and eccentricity, shew how to describe the curve.

10. Having given the focus and three points of a conic, shew how to describe the curve.

11. If PN be the principal ordinate of any point P on a conic, then

$$SP \pm SL : SN = \text{the eccentricity},$$

where SL is the semi-latus rectum.

12. Determine the condition that a chord of a conic may be greater than, equal to, or less than the diameter of the eccentric circle of its middle point.

13. If the point p (Art. 6) describe a series of circles about the same centre O, the point P will describe a series of conics having the same focus and directrix; and the eccentricities of the conics will be to one another as the radii of the circles.

14. If pm be drawn perpendicular to the directrix (Art. 6), then $pm \,.\, PM = OD \,.\, SX$. Hence shew that every focal chord is divided harmonically by the focus and directrix.

15. At any point of a conic the tangent makes a greater or less angle with the focal distance than with the perpendicular to the directrix according as the eccentricity is a ratio of minority or of majority.

16. If two conics have a common focus, their common chord or chords will pass through the point of concourse of their directrices.

17. If they also touch each other, and if from any point on their common tangent second tangents be drawn to them, the line joining their points of contact will pass through the focus.

18. The tangents at the ends of a focal chord meet the latus rectum at points equidistant from the focus.

19. The focal distance of any point on a conic is equal to the ordinate of the point produced to meet the tangent at an extremity of the latus rectum.

20. If a chord of a conic subtends a constant angle at the focus, its envelope and the locus of its pole are conics having the same focus and directrix, and the eccentricities of the three are proportionals.

21. If two tangents TP, TQ to a conic meet any third tangent in p and q, the angle pSq will be equal or supplementary to $\frac{1}{2}PSQ$.

22. The vertex of a circumscribed triangle whose base subtends a constant angle at the focus is a conic.

23. If TP be a tangent from P to a conic, and if the ordinate of T meet the curve in Q, the projection of ST upon SP is equal to SQ.

24. Shew that PN and QM in Art. 15 meet on the axis.

25. Given the focus of a conic inscribed in a triangle, find the points of contact.

26. Prove that the line PR in Art. 8 is a tangent according to Euclid's definition. Also state Prop. II. in a form which will still be applicable if PR be any *secant* through P.

27. Given the focus of a conic and a focal chord, the locus of the extremities of the latus rectum is a circle.

28. Given a chord of a conic and the angle which it subtends at the focus, the focal distance of the intersection of the tangents at its extremities passes through a fixed point.

29. The portion of any tangent to a conic intercepted between the tangents at the ends of the parallel focal chord is divided by its point of contact into segments equal to the distance of that point from the focus, and subtends a right angle at the focus.

30. If the tangent at any point of a conic meet the directrix in D and the latus rectum in L, then

$$SL : SD = SA : AX.$$

31. If M be the projection upon the directrix of any point P on a conic, shew that SM meets the tangent at the vertex upon the bisector of the angle SPM.

32. Two sides of a triangle being given in position, if the third subtends a constant angle at a fixed point determine its envelope.

33. If SY be the focal perpendicular on the tangent at P, and X the foot of the directrix, then

$$SY : YX = SA : AX.$$

Determine the locus of Y, and shew that it is the envelope of the circle on SP as diameter.

34. Shew also that XY meets the latus rectum at the foot of the perpendicular to it from P.

35. If the diameter at a point P on a conic bisects the chord normal at Q, the diameter at Q bisects the chord normal at P.

36. The normal PG (Art. 11) becomes equal to the semi-latus rectum when P coalesces with the vertex.

37. The perpendicular from G to SP varies as the ordinate of P; and the line joining the foot of this perpendicular to the foot of the ordinate of P is parallel to SM.

38. If Q be any point on the normal at P, and if QL be a perpendicular to SP, and QM a perpendicular to the ordinate of P, then

$$QL : PM = SA : AX.$$

39. The perpendicular to any focal chord from the intersection of the normals at its extremities meets the chord at a distance from one extremity equal to the focal distance of the other; and the parallel to the axis from the intersection of the normals bisects the chord.

40. If P be the intersection of the tangents at the extremities of any normal chord of a conic, and Q the point in which it meets the directrix, the circle SPQ passes through an extremity of the chord.

41. A circle which touches a conic and has its centre upon the axis intercepts a constant length upon the focal radius to either point of contact.

42. If QQ' be the focal chord at right angles to the normal PG, then

$$PG^2 = SQ . SQ'.$$

43. Construct a conic of which an arc is given.

44. The parallel diameters of any two similarly situated conics of the same eccentricity bisect the same systems of parallel chords. If the conics be also concentric, the lengths which they intercept on any chord are equal two and two.

45. The angle between any two chords of a conic is equal to the angle subtended at the focus by the length which the conjugate diameters intercept on the directrix.

46. Any chord of a conic and the conjugate diameter meet the axis and the directrix respectively upon a line parallel to the focal distance of the pole of the chord (Art. 69). In a central conic the polar of any point meets the axis at a distance from the centre which varies inversely as the central abscissa of the point.

47. The triangle whose vertices are the focus of a conic and the two points in which any tangent thereto and the conjugate diameter meet the axis and the directrix respectively, has its orthocentre at the point in which the tangent meets the directrix.

48. Any focal chord of a conic and its diameter meet the directrix at points whose ordinates contain a constant rectangle. Determine the locus of the middle point of a focal chord.

49. Any two tangents to a conic intercept upon a tangent drawn parallel to their chord of contact a length which is bisected at the curve.

50. If two chords AB, CD of a conic (not being parallel to one another) make equal angles with the axis, then will AC, BD and likewise AD, BC make equal angles with the axis.

51. If a chord of a conic subtends equal angles at the extremities of another chord, it subtends equal angles at the extremities of any chord parallel to the latter.

52. If a circle be drawn touching a conic at P and cutting it at Q and R, then will QR and the tangent at P be equally inclined to the axis. Shew how to draw a circle touching a conic at any given point (other than the vertex) and also cutting it at the same point.

The Parabola.

53. If the ordinates of a parabola be cut in a constant ratio the points of section will lie on a parabola.

54. Circles whose radii are in arithmetical progression touch a given straight line on the same side at a given point. If to each circle a tangent parallel to the given line be drawn, it will cut the circle next larger in points lying on a parabola.

55. Deduce from Art. 19 that the middle points of all parallel chords in the parabola are at the same distance from the axis.

56. If an equilateral triangle circumscribes a parabola, the focal radii to its vertices pass each through the opposite point of contact.

57. A focal chord being drawn to meet the tangent at a constant angle, determine the locus of their intersection.

58. The circle on a chord of a parabola as diameter does not meet the directrix unless the chord passes through the focus.

59. In Art. 20 shew that $\angle QOQ' = MYM'$. In what case are these angles right angles?

60. The perpendicular to a chord of a parabola from its middle point V meets the axis at a distance equal to SX from the foot of the ordinate of V.

61. A point within a parabola is nearer to the focus than to the directrix.

62. Shew that all parabolas are similar curves.

63. Prove the following construction. Take any ordinate NP, and draw PM parallel and equal to NA. Divide NP into any number of equal parts, and through the points of section draw parallels p_1, p_2, p_3,... to the axis. Divide MP into the same number of equal parts in points 1, 2, 3,... Then the lines p_1, p_2, p_3,... meet $A1$, $A2$, $A3$,... respectively on the parabola.

64. The semi-latus rectum is a mean proportional between the principal ordinates of the ends of a focal chord. And if AM, AM' be the corresponding abscissæ, then $AM.AM' = AS^2$.

65. In Art. 22 shew that

$$QD^2 = 4AS.PV.$$

66. A circle can be described touching any two diameters of a parabola and the focal radii to their extremities.

67. The locus of the centre V of a focal chord is another parabola. Prove this by shewing that if VL be the ordinate of V, then

$$VL^2 = 2AS.AL.$$

68. A chord QQ' is cut in O by a diameter which meets the curve in P. Shew that if R be a point on the curve whose abscissa is PO, and PV, PV' be the abscissæ of Q, Q', then

$$QV^2 - Q'V'^2 : QV^2 - OR^2 = QV + Q'V' : QV.$$

Deduce that $QV.Q'V'$ is equal to OR^2, and $PV.PV'$ to PO^2.

69. Any triangle whose base is parallel to the axis of a parabola has its remaining sides in the subduplicate ratio of the parallel focal chords.

70. Shew how to place in a parabola a focal chord of given length.

71. A parabola being given find its axis and focus.

72. If PQ be a chord which subtends a right angle at A, and AN, AM be the principal abscissæ of P, Q, then PQ passes through a fixed point, and

$$AN.AM = PN.PM = 16AS^2.$$

73. Given the directrix of a parabola and two points on the curve, two positions of the focus can be determined; or given the

focus and two points, two positions of the directrix can be determined.

74. If two parabolas have a common focus, their common chord passes through the intersection of their directrices and bisects the angle between them.

75. If two parabolas have a common directrix, their common chord bisects the straight line joining their foci at right angles.

76. The straight lines which join the ends of a focal chord to the vertex meet the directrix on the diameters through the ends of the chord.

77. Find the locus of the centre of a circle which passes through a given point and touches a given straight line; or which touches a given circle and a given straight line.

78. Circles being described on the segments of a focal chord as diameters, the straight line joining their centres subtends right angles at the intersections of their common tangents.

79. If a circle cuts a parabola in points 1, 2, 3 above the axis and in a point 4 below it, the difference of the ordinates of 1, 3 is to the difference of their abscissæ as the sum of the ordinates of 2, 4 to the difference of their abscissæ. Deduce that the ordinate of 4 is equal to the sum of the ordinates of 1, 2, 3.

Examine the case in which 1, 2, 3 coalesce.

80. If 1, 2 and 3, 4 lie on opposite sides of the axis the sum of the ordinates of 1, 2 is equal to the sum of the ordinates of 3, 4.

81. If a leaf of a book be folded so that one corner moves along an opposite side the direction of the crease touches a parabola.

82. If from any point T on a fixed tangent a second tangent TP be drawn, the angle STP will be constant.

83. PQR being a circumscribed triangle, the perpendiculars from P, Q, R to SP, SQ, SR cointersect.

84. If the tangents at P, Q meet in T, and if C be the centre of the circle TPQ, then CST is a right angle.

85. Two parabolas which have a common focus and their axes in opposite directions intersect at right angles.

86. The perpendicular drawn to a normal from the point in which it meets the axis envelopes an equal parabola.

87. The normal at any point is equal to twice the focal perpendicular upon the tangent, and is also a mean proportional between the focal distance of that point and the latus rectum.

88. The squares of the normals at the ends of a focal chord are together equal to the square of twice the normal perpendicular to the chord.

89. The circle on a focal radius touches the tangent at A.

90. The diameter through one end of a focal chord bisects the chord normal at the other.

91. The locus of the foot of the focal perpendicular on the normal is a parabola.

92. The tangent at any point meets the directrix and the latus rectum in points equidistant from the focus.

93. A normal chord of a parabola produced to meet the directrix subtends a right angle at the pole of the chord.

94. The normal at any point is equal to the ordinate which bisects the subnormal at that point.

95. The tangents at right angles drawn to two parabolas which have a common focus and axis meet on a fixed straight line.

96. If one triangle can be inscribed in a given circle so that its three sides touch a given parabola, any number of triangles can be so described.

97. A triangle revolves about its vertex in one plane: prove that at any instant the directions of motion of all the points of its base are tangents to a parabola.

98. Tangents being drawn to a parabola from any point T, the diameters through their points of contact meet any secant PQ which passes through T in M and N: shew that

$$TM^2 = TN^2 = TP \, . \, TQ.$$

99. Given two chords of a parabola, find the direction of its axis, and shew that there are two solutions.

100. If two tangents to a parabola be cut by any third, their alternate segments have the same ratio, and this ratio is constant when the two tangents are fixed.

101. If a circle and a parabola touch and cut one another at the same point (Ex. 52), their common chord is equal to four

times their common tangent at that point, terminated by the axis.

102. A diameter meeting a chord and the tangent at an end of it is cut by the curve in the ratio in which it cuts the chord.

103. Draw a chord which shall be cut in a given ratio by a given diameter.

104. The intercepts on any diameter of a parabola between any two tangents and the ordinates of their points of contact to that diameter are equal; and the triangle contained by the two tangents and the diameter is equal to half the trapezium bounded by their chord of contact, the two ordinates and the diameter.

105. Three tangents to a parabola form a triangle equal to half the triangle determined by their points of contact.

Central Conics.

106. In Art. 33 shew that $Z'Ap$ and ZpA' are straight lines.

107. The sides AD, DC of a rectangle $ABCD$ are divided into the same number of equal parts, and straight lines are drawn from B, A respectively to the points of section. Shew that corresponding lines in the two series meet on an ellipse whose axes are equal to the sides of the rectangle.

108. A parallelogram $ABCD$ has its diagonal AC at right angles to the side AB. If CD be divided into any number of equal parts and straight lines be drawn from A to the points of section, and if AC be divided into the same number of equal parts and straight lines be drawn from B to the points of section, then will corresponding lines in the two series meet on a hyperbola.

109. Two ellipses whose major axes are equal have a common focus; prove that they intersect in two points only.

110. Given one focus of a central conic, a point on the curve, and the length of the axis; find the locus of the further focus, and the locus of the centre.

111. What is the locus of the centre of a circle which touches two fixed circles?

112. Given a central conic, find its centre and foci.

113. A bifocal conic being defined as the locus of a point the sum or difference of whose distances from two fixed points is constant, prove the property of the directrices and the property of the principal ordinate[*]. Also shew that the sum or difference of the focal distances of any point without the conic is greater than the axis, and conversely.

114. If P be any point on a conic whose foci are S and S', the circles on SP, $S'P$ and the transverse axis as diameters have their radical centre on the ordinate of P.

115. The circle inscribed in the triangle SPS' touches SP in M, and SS' in N. Prove that $PM = A'S$, and $AM = SP$.

116. A diameter of an ellipse varies inversely as the perpendicular focal chord of the auxiliary circle.

117. The ordinates to the axes at the points in which a common diameter meets the major and minor auxiliary circles of an ellipse intersect two and two on the ellipse.

118. A point in a straight line which slides between two fixed straight lines at right angles traces an ellipse.

119. A circle can be drawn through the foci and the intersections of any tangent with the tangents at the vertices.

120. Any diameter is divided harmonically by a double ordinate and the point of concourse of the tangents at its extremities.

121. The exterior angle between any two tangents to an ellipse is an arithmetic mean to the angles which either tangent subtends at the two foci.

122. The focal radii to the two ends of a diameter make equal angles with the tangents thereat.

123. The ordinate PN bisects the angle YNY' (Art. 39), and the points $YNCY'$ are concyclic.

124. Also $SY^2 : CB^2 = SP : 2CA \pm SP$.

125. Also, if CD be the radius conjugate to CP,

$$SY : SP = CB : CD.$$

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[*] In the ellipse, if $SP + S'P = 2CA$ and CN be the abscissa, it may be shewn that $SP^2 \sim S'P^2$ is equal to $4CS \,.\, CN$, and $SP \sim S'P$ to $2e \,.\, CN$ (where e stands for the ratio $CS : CA$). It may also be shewn (Lemma D) that the *sum of* $(SP + S'P)^2$ and $(SP \sim S'P)^2$ is equal to $4(CS^2 + CN^2 + PN^2)$.

126. The normal at P is a harmonic mean to SY, $S'Y'$ (Art. 39), and is bisected by $S'Y$ and SY'.

127. Tangents being drawn from any point on a circle through the foci, shew that the bisectors of the angles between them pass through fixed points.

128. If the tangent and normal meet either axis in T, G, then

$$CG \, . \, CT = CS^2.$$

129. The bisectors of the angles between the tangents from any point are tangent and normal to the confocals through that point.

130. The pole of the tangent at P with respect to the auxiliary circle lies on the ordinate of P.

131. A circumscribing parallelogram which has two corners on the directrices has the other two on the auxiliary circle.

132. If an ellipse inscribed in a triangle has one focus at the orthocentre, the other focus will be at the centre of the circumscribed circle.

133. If an ellipse slides between two straight lines at right angles the locus of its centre is a circle.

134. Tangents at right angles intersect on a fixed circle, the square of whose radius is $CA^2 \pm CB^2$. Explain the result when $CA^2 - CB^2$ vanishes, or is negative.

135. The straight line joining the foci subtends at the pole of a chord half the sum or difference of the angles which it subtends at the extremities of the chord.

136. The portion of a normal chord intercepted between the directrices subtends at the pole of the chord half the sum of the angles which the straight line joining the foci subtends at the extremities of the chord (Ex. 40).

137. If a chord be produced to meet the directrices, the parts produced will subtend equal angles at the pole of the chord.

138. Every ellipse has one pair of equal conjugate diameters, and they coincide with the diagonals of the rectangle formed by the tangents at the extremities of the axes. Has the hyperbola any corresponding property?

139. If CP and CD be conjugate radii of an ellipse,

$$(SP - CA)^2 + (CA - SD)^2 = CS^2.$$

140. When is the sum or difference of conjugate diameters greatest, and when least?

141. The tangent from N to the circle on XX' (Fig. Art. 49) as diameter varies as the normal at P, and the tangent to the auxiliary circle varies as PN.

142. If N be a point in AA' produced the circles described about S, S' with radii AN, $A'N$ meet on the hyperbola. What is the corresponding construction for the ellipse?

143. If PL, PL' be supplemental chords of an ellipse equally inclined to the tangent at P, then $PL + PL'$ is equal to the diameter of the circle on which tangents at right angles intersect (Ex. 134).

144. From a fixed point O, OP is drawn to a given circle. Find the envelope of a straight line through P inclined at a constant angle to OP.

145. In a central conic a circle through P and either G or g cuts off from the focal distances lengths whose sum is constant.

146. Given in an ellipse a focus and two points, the other focus describes a hyperbola.

147. If P, Q be points on a central conic a confocal passes through the intersections of SP, $S'Q$ and SQ, $S'P$.

148. The tangents at these points and at P, Q cointersect.

149. If PP', DD' be conjugate diameters of a hyperbola and Q any point on the curve, shew that $QP^2 + QP'^2$ exceeds $QD^2 + QD'^2$ by a constant quantity.

150. Given two points of a parabola and the direction of its axis, the locus of the focus is a hyperbola.

151. A chord which subtends a right angle at the vertex meets the axis in a fixed point.

152. If a hyperbola touches the sides of a quadrilateral inscribed in a circle and if one focus lies on the circle, the other lies on the circle.

153. The triangle whose base is equal to the transverse axis, and its remaining sides to the focal distances of any external point, has its vertical angle equal to the angle between the tangents to the conic from that point and its remaining angles to the angles which either tangent subtends at the foci.

154. The projection of the normal at any point, terminated by the conjugate axis, upon either focal distance is equal to the transverse semi-axis.

155. If AM and $A'M$ be taken on the axis equal to the focal distances of any point P on the ellipse, then

$$CP^2 = CB^2 + CM^2.$$

Deduce the property of the principal ordinate.

156. If two ellipses having equal axes be placed vertex to vertex, and one of them then roll upon the other, either of its foci will describe a circle about a focus of the latter.

157. The common diameters of equal similar and concentric ellipses are at right angles.

158. The diagonals of any parallelogram circumscribing a conic are conjugate diameters; and the sides of any inscribed parallelogram are parallel to conjugate diameters.

159. The sum or difference of the reciprocals of the squares of any two diameters at right angles is constant.

160. Determine the positions of a chord of an ellipse which subtends right angles at both foci.

161. The opposite sides of a quadrilateral described about an ellipse subtend supplementary angles at either focus.

162. A circumscribed quadrilateral whose diagonals meet at the centre of the conic must be a parallelogram.

163. If P, Q be points on a conic, CM and CN their abscissæ, and T the point in which PQ meets the axis, then

$$CT\,(PM - QN) = PM \,.\, CN - QN \,.\, CM.$$

164. If CP, CD and CP', CD' be conjugate radii, and if PN, DR be ordinates to CP', then

$$CN^2 \pm CR^2 = CP'^2 ; \quad PN^2 \pm DR^2 = CD'^2 ;$$

and $$PN : CR = DR : CN = CD' : CP'.$$

165. The vertices of the conjugate parallelograms of an ellipse lie on a similar ellipse, and their polars envelope a similar ellipse. What are the corresponding properties of the hyperbola?

166. The inscribed parallelogram whose diameters are at right angles envelopes a circle, the reciprocal of the square of whose radius is equal to $\frac{1}{CA^2} \pm \frac{1}{CB^2}$.

167. With the orthocentre of a triangle as centre two ellipses are described, the one touching its sides and the other passing through its vertices: prove that they are similar, and that their homologous axes are at right angles.

168. If two ellipses having equal major axes be inscribed in a parallelogram, their foci determine an equiangular parallelogram.

169. Every circular arc which terminates at two given points is trisected by a pair of fixed hyperbolas, whose eccentricity is equal to 2.

170. The circle to which any such arc belongs cuts either hyperbola at one of the given points, and at three other points which determine an equilateral triangle.

171. The tangent at P meets two conjugate diameters in T, t; and TS, tH meet in Q; prove that the triangles SPT, HPt, TQt are similar.

172. The two points on a central conic at which any chord subtends the greatest and least angles are at the ends of a diameter equal to that which bisects the chord.

173. The parallelograms whose diagonals are any two diameters and their conjugates respectively are equal.

174. A circle being drawn through the foci of an ellipse, their common tangents touch the circle upon the tangent to the ellipse at one end of its minor axis.

175. The tangent to a conic at a given point meets any two parallel tangents in points whose focal distances meet on a fixed circle, having its centre on the normal at the given point.

The Asymptotes.

176. Given the asymptotes of a hyperbola and a point on the curve, determine the foci and directrices.

177. At any point P of a hyperbola SP is equal to a line drawn parallel to an asymptote to meet the directrix.

178. Given a pair of conjugate diameters, two hyperbolas can be described: these have the same asymptotes, and every two diameters conjugate in the one are conjugate in the other*.

179. If two hyperbolas have the same asymptotes a chord of one touching the other is bisected at the point of contact.

180. The straight lines joining the points in which two tangents meet the asymptotes are parallel.

181. A hyperbola can be drawn through the ends of any two radii of an ellipse so as to have the conjugate diameters as asymptotes.

182. Find the locus of a point which divides the part of any tangent between the asymptotes in a constant ratio.

183. The asymptotes of a hyperbola may be regarded as tangents whose points of contact are at infinity.

184. If the ordinate at P to either axis meets the nearer asymptote in E, the perpendicular through E to the asymptote passes through the point in which the normal at P meets that axis.

185. Any tangent and its normal meet the asymptotes and the axes respectively in four points concyclic with the centre.

186. The intercept made by the directrices of a hyperbola on either asymptote is equal to the transverse axis.

187. The point of contact of the tangent from the intersection of an asymptote with a directrix lies upon a focal radius parallel to that asymptote.

188. The intercept on any tangent between the asymptotes subtends at the further focus an angle equal to half the angle between them: it also subtends a constant angle at the intersection of its normal with either axis.

189. The chords of intersection of any circle with the asymptotes are equally inclined to the axis.

190. The products of the segments of any two intersecting chords of the asymptotes are as the parallel focal chords of the hyperbola.

191. Any circle which touches both branches makes an intercept equal to the axis upon either asymptote.

---

* Two hyperbolas thus related are said to be conjugate.

192. The lines connecting the extremities of any pair of focal chords parallel to conjugate diameters meet on the asymptotes.

193. The asymptotes and any two conjugate diameters divide any straight line harmonically.

194. The lines joining a variable point on a hyperbola to two fixed points upon it intercept a constant length on either asymptote.

195. The axes of the two parabolas which have a common focus and pass through two given points are parallel to the asymptotes of the hyperbola which passes through their focus and has the given points for foci.

196. If the abscisses on either asymptote of any number of points on a hyperbola are in arithmetical progression, their ordinates are in harmonical progression.

197. The area of the hyperbolic sector determined by any two radii CP, CQ is equal to the area cut off from the space between the asymptotes by the parallels from P and D to either asymptote.

198. The line joining a pair of adjacent extremities of any two conjugate diameters is parallel to one asymptote and is bisected by the other.

199. Determine the locus of the centroid of the triangle formed by the asymptotes and any tangent.

200. Supposing the axes of a hyperbola to vanish whilst its eccentricity remains unaltered, determine the limiting form of the curve.

The Equilateral Hyperbola.

201. The nine-point circle of any inscribed triangle passes through the centre of the curve.

202. Tangents to a parabola which include the supplement of half a right angle intersect on a rectangular hyperbola.

203. Every right-angled triangle inscribed in an equilateral hyperbola has its hypotenuse parallel to the normal at the opposite angle: hence shew how to draw the tangent at any given point on the curve.

204. The base of a triangle and the sum or difference of its base-angles being given, the locus of its vertex is an equilateral conic.

205. The locus of the middle point of a straight line which cuts off a constant area from a corner of a square is an arc of a rectangular hyperbola.

206. The bisectors of the angles between supplemental chords of an equilateral hyperbola are parallel to the asymptotes.

207. Diameters at right angles bisect chords at right angles; and any chord subtends equal or supplementary angles at the ends of a perpendicular chord.

208. If on an arc AB of a circle whose centre is O there be taken points P, Q such that arc $AP = 2$ arc BQ, then a rectangular hyperbola described on AO as diameter so as to pass through the intersection of OB with the tangent to the circle at A will also pass through the intersection of AP, OQ.

209. Shew also that the hyperbola and the completed circle intersect in three points (other than A) which determine an equilateral triangle; and deduce that the problem *to trisect a given angle* admits of three solutions.

210. If two concentric rectangular hyperbolas be such that the axes of one are the asymptotes of the other, they will intersect at right angles.

211. A circle through the centre and two points of a rectangular hyperbola passes also through the intersection of the lines drawn from each of the two points parallel to the polar of the other.

212. CY being drawn perpendicular to the tangent at P, the triangles PCA, CAY are similar.

213. If CP, CD be conjugate radii of a rectangular hyperbola, then will D be the reflexion of P with respect to one of the asymptotes.

214. Ellipses being inscribed in a parallelogram, their foci lie on a rectangular hyperbola.

215. A conic through the four common points of two rectangular hyperbolas is itself a rectangular hyperbola.

216. A conic through the centres of the four circles which touch the sides of a triangle is a rectangular hyperbola, and its centre is on the circumscribing circle.

217. On opposite sides of a chord of a rectangular hyperbola equal segments of circles are described. Shew that the four points in which the circles meet the curve again are the vertices of a parallelogram.

218. The tangents to a rectangular hyperbola at the vertices of an inscribed triangle meet two and two on the lines joining the feet of the perpendiculars of the triangle.

219. If each vertex of a triangle be the pole of the opposite side with respect to an equilateral hyperbola, the circumscribing circle will pass through the centre of the hyperbola.

220. A circle and an equilateral hyperbola intersect in four points: if one of their common chords is a diameter of the hyperbola, the other is a diameter of the circle.

221. The circles described upon the six common chords of any two equilateral hyperbolas as diameters cut one another orthogonally in opposite pairs.

222. Of two chords at right angles or conjugate in direction one and one only is a chord of a single branch.

223. If two focal chords be conjugate in direction, the lines joining their extremities pass through fixed points on the asymptotes.

224. Any two conjugate semi-diameters contain equal and similar triangles with the ordinates and abscissæ of their extremities to any other diameter.

225. The opposite arcs cut off by any two diameters subtend equal angles at any point on the curve.

The Cone.

226. In Art. 59 shew that $AS = A'H$, and $SH = OA \pm OA'$.

227. A conic section may be regarded as the locus of a point the sum or difference of whose distances from a point in its plane and a point without it is constant.

228. Express the eccentricity of a section of a cone in terms of the angles which the axis of the cone makes with its sides and with the axis of the section.

229. The sections by identical planes of the cones touching *two given spheres* have their eccentricities in a constant ratio.

230. The sphere having for diameter the line joining the centres of the focal spheres of a section contains its auxiliary circle.

231. At any point of a section the tangent makes equal angles with the focal distances and the side of the cone.

232. If T be any point on the tangent at P to a section, the two tangents to it from T are inclined at the same angle as the tangents TQ, TR (Art. 59) to the spheres.

233. The conjugate axis of a section is a mean proportional to the diameters of its focal spheres.

234. The perpendiculars upon any tangent to a section from the centres of its focal spheres are at right angles to one another.

235. The latus rectum of any section whose plane touches a sphere about the vertex of the cone as centre is equal to the diameter of the circular sections whose planes touch the sphere.

236. The asymptotes of a hyperbolic section are parallel to the sides of the cone which lie on the parallel plane through its vertex.

237. Shew how to cut a section of maximum eccentricity from a given cone.

238. A conic section may be regarded as the locus of a point the sum or difference of the tangents from which to two fixed circles is constant.

Miscellaneous.

239. The lines joining the extremities of any two focal chords of a conic intersect on the directrix, and the focal distances of their intersections are at right angles.

240. If PQ be a chord of a conic, and if the parallel focal chord F meet the tangent at P in T, then

$$PQ \, . \, ST = F \, . \, SP.$$

241. Prove that the perpendiculars from any point on a circle to a fixed chord and to the tangents at its extremities are continued proportionals. What is the corresponding property of the ellipse?

242. The focal distances of g (Art. 42) meet the directrices upon the parallel to the axis through P.

243. One triangle being inscribed and another circumscribed to a parabola, if their sides be parallel each to each they will be in the ratio of four to one.

244. The area of a parabolic segment is to a triangle of the same base and altitude as four to three.

245. Given an ellipse, draw a pair of conjugate diameters inclined at a given angle; and determine the axes and foci.

246. A variable tangent to one conic which cuts off a constant area from another is bisected by the former.

247. If TP, TQ be tangents to an ellipse and CP', CQ' the parallel radii, shew that the triangles TPQ and $CP'Q'$ are together equal to the trapezium $CPTQ$, and likewise to the triangle whose sides are equal to ST, HT, AA' (Ex. 153).

248. Prove also that

$$TP \, . \, TQ + CP' \, . \, CQ' = TS \, . \, TH.$$

249. Every central conic through four points has a pair of conjugate diameters parallel to the axes of the two parabolas through the same four points (Ex. 99).

250. No two conics can meet one another in more than four points.

THE END.

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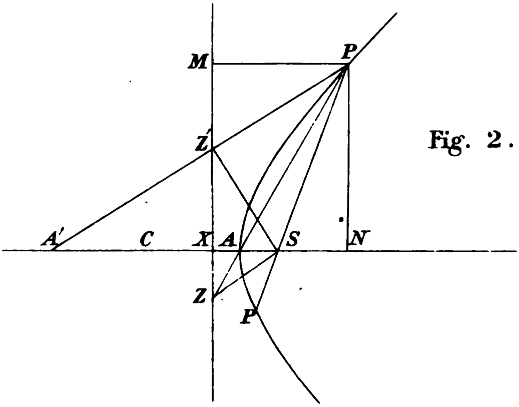


Fig. 2.

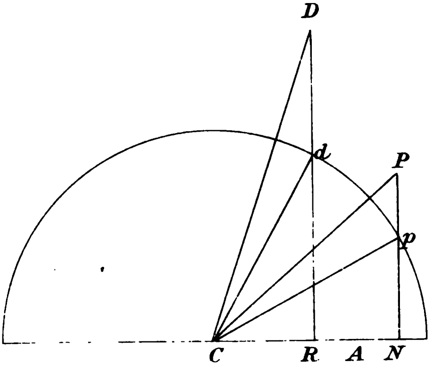
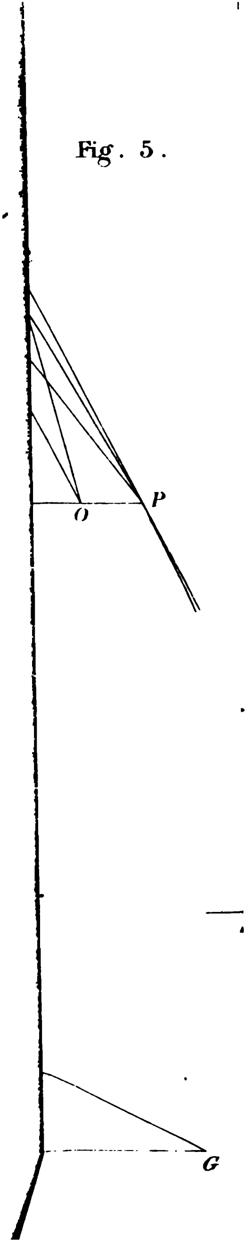


Fig. 3.

Fig. 5.



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